ON REGULARITY OF CONJUGACY BETWEEN LINEAR COCYCLES OVER PARTIALLY HYPERBOLIC SYSTEMS

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Abstract. We consider Hölder continuous $GL(d,\mathbb{R})$-valued cocycles, and more generally linear cocycles, over an accessible volume-preserving center-bunched partially hyperbolic diffeomorphism. We study the regularity of a conjugacy between two cocycles. We establish continuity of a measurable conjugacy between *any* constant $GL(d,\mathbb{R})$-valued cocycle and its perturbation. We deduce this from our main technical result on continuity of a measurable conjugacy between a fiber bunched linear cocycle and a cocycle with a certain block-triangular structure. The latter class covers constant cocycles with one Lyapunov exponent. We also establish a result of independent interest on continuity of measurable solutions for twisted vector-valued cohomological equations over partially hyperbolic systems. In addition, we give more general versions of earlier results on regularity of invariant subbundles, Riemannian metrics, and conformal structures.

1. Introduction and main results

Cocycles and their cohomology play an important role in dynamics. In this paper we consider $GL(d,\mathbb{R})$-valued cocycles, and more generally linear cocycles, over a volume-preserving partially hyperbolic diffeomorphism f of a compact manifold $\mathcal{M}$. The prime examples are given by the differential of f and its restrictions to invariant subbundles, for example stable, unstable, or center. Such cocycles are used in the study of dynamics and rigidity of hyperbolic and partially hyperbolic systems.

First we discuss $GL(d,\mathbb{R})$-valued cocycles.

Definition 1.1. *Let $A : \mathcal{M} \to GL(d,\mathbb{R})$ be a continuous function. The $GL(d,\mathbb{R})$-valued cocycle over f generated by A is the map $\mathcal{A} : \mathcal{M} \times \mathbb{Z} \to GL(d,\mathbb{R})$ defined as follows: for $x \in \mathcal{M}$ and $n \in \mathbb{N}$,*

$$\mathcal{A}_x^0 = Id, \quad \mathcal{A}_x^n = A(f^{n-1}x) \circ \cdots \circ A(x) \quad \textit{and} \quad \mathcal{A}_x^{-n} = (\mathcal{A}_{f^{-n}x}^n)^{-1}.$$

If the tangent bundle of $\mathcal{M}$ is trivial, $T\mathcal{M} = \mathcal{M} \times \mathbb{R}^d$, then the differential Df can be viewed as a $GL(d,\mathbb{R})$-valued cocycle with $A(x) = Df_x$ and $\mathcal{A}_x^n = Df_x^n$.

A natural equivalence relation for cocycles is defined as follows.

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Definition 1.2. *Two $GL(d,\mathbb{R})$-valued cocycles $\mathcal{A}$ and $\mathcal{B}$ over f are continuously (resp. μ-measurably, with respect to a measure μ on $\mathcal{M}$)* cohomologous *if there exists a continuous (resp. μ-measurable) function $C : \mathcal{M} \to GL(d,\mathbb{R})$ such that*

$$\mathcal{B}_x = C(fx) \circ \mathcal{A}_x \circ C(x)^{-1} \text{ for all } x \in \mathcal{M} \text{ (resp. } \mu\text{-almost everywhere).} \tag{1.1}$$

We refer to C as a continuous (resp. μ-measurable) conjugacy *between $\mathcal{A}$ and $\mathcal{B}$.*

We consider the question whether a measurable conjugacy between two cocycles is continuous. A positive answer was obtained by Wilkinson in [W13] for Hölder continuous $\mathbb{R}$-valued cocycles over an accessible center-bunched volume preserving C^2 partially hyperbolic diffeomorphism.

For cocycles with values in non-commutative groups, studying cohomology is more difficult. Usually additional assumptions related to their growth are made, such as fiber bunching. The latter means that non-conformality of the cocycle is dominated by the expansion and contraction in the base, see Definition 2.2. The first result on continuity of a measurable conjugacy for non-commutative cocycles over partially hyperbolic systems was obtained in [KS16, Theorem 4.2]. It extended earlier results for cocycles over hyperbolic diffeomorphisms [Sch99, NP99, PW01, S15]. There we established continuity of a measurable conjugacy between Hölder continuous fiber bunched cocycles, one of which is uniformly quasiconformal. A cocycle $\mathcal{A}$ is *uniformly quasiconformal* if $\|\mathcal{A}_x^n\| \cdot \|(\mathcal{A}_x^n)^{-1}\|$ is uniformly bounded in $x \in \mathcal{M}$ and $n \in \mathbb{Z}$.

In contrast to scalar cocycles, a measurable conjugacy between $GL(d,\mathbb{R})$-valued cocycles is not always continuous, even when f is hyperbolic and both cocycles are close to the identity. Indeed, in [PW01] Pollicott and Walkden constructed smooth $GL(2,\mathbb{R})$-valued cocycles over an Anosov toral automorphism of the form

$$\mathcal{A}_x = \begin{bmatrix} a(x) & b(x) \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathcal{B}_x = \begin{bmatrix} a(x) & 0 \\ 0 & 1 \end{bmatrix} \tag{1.2}$$

that are measurably (with respect to the Lebesgue measure m), but not continuously cohomologous. We note that these cocycles have two Lyapunov exponents, 0 and $\int \log a(x)\, dm < 0$. Thus in general one can not expect continuity of a measurable conjugacy in case of more than one Lyapunov exponent.

The next theorem gives a positive result for a constant cocycle $\mathcal{A}$ with one Lyapunov exponent, which means that all eigenvalues of the matrix A generating $\mathcal{A}$ have the same modulus.

Assumption 1.3. *In this paper, f is an accessible center-bunched C^2 partially hyperbolic diffeomorphism of a compact manifold $\mathcal{M}$ preserving a volume μ. (See Section 2.1 for details.)*

Theorem 1.4. *Let (f,μ) be as in Assumption 1.3. Let $\mathcal{A}$ be a constant $GL(d,\mathbb{R})$-valued cocycle with one Lyapunov exponent and let $\mathcal{B}$ be an su-β-Hölder fiber bunched $GL(d,\mathbb{R})$-valued cocycle over f. Then any μ-measurable conjugacy between $\mathcal{A}$ and $\mathcal{B}$ coincides μ-a.e. with an su-β-Hölder conjugacy.*

We say that a function is su-β-Hölder if it is continuous on $\mathcal{M}$ and β-Hölder continuous along the leaves of stable and unstable manifolds for f, see Section 2.3.

As a corollary of Theorem 1.4, we obtain continuity of a measurable conjugacy between *any* constant $GL(d,\mathbb{R})$-valued cocycle and its perturbation, without fiber bunching or one Lyapunov exponent assumptions on either cocycle.

Theorem 1.5. *Let (f,μ) be as in Assumption 1.3. Let $\mathcal{A}$ be a constant $GL(d,\mathbb{R})$-valued cocycle over f. Then for any Hölder continuous $GL(d,\mathbb{R})$-valued cocycle $\mathcal{B}$ sufficiently C^0 close to $\mathcal{A}$, any μ-measurable conjugacy between $\mathcal{A}$ and $\mathcal{B}$ coincides μ-a.e. with an su-Hölder conjugacy.*

We deduce Theorem 1.4 from a more general result, Theorem 1.6 below. It holds in a broader context of linear cocycles on vector bundles, see Section 2.4 for details. Also, instead of a constant cocycle with one exponent we consider a cocycle with a certain "block-triangular" structure. As we show in Proposition 4.5, this structure implies that the cocycle is fiber bunched and has one Lyapunov exponent for each f-invariant ergodic measure. For a hyperbolic f, the converse also holds by [KS13, Theorem 3.9]. However, the converse is not known and may not hold in general in the partially hyperbolic case, where existing results, such as [KS13, Theorem 3.4], give a weaker structure.

We say that a linear cocycle $\tilde{\mathcal{A}}$ on a vector bundle $\tilde{\mathcal{E}}$ over $\mathcal{M}$ is *uniformly bounded* if $\|\tilde{\mathcal{A}}_x^n\|$ is uniformly bounded in $x \in \mathcal{M}$ and $n \in \mathbb{Z}$. This notion does not depend on the choice of a continuous norm on $\tilde{\mathcal{E}}$.

Theorem 1.6. *Let (f,μ) be as in Assumption 1.3. Let $\mathcal{E}$ and $\mathcal{E}'$ be β-Hölder vector bundles over $\mathcal{M}$, or more generally su-β-Hölder subbundles of β-Hölder vector bundles over $\mathcal{M}$. Let $\mathcal{A}$ be an su-β-Hölder linear cocycle on $\mathcal{E}$ over f. Suppose that there exist a flag of su-β-Hölder $\mathcal{A}$-invariant sub-bundles*

$$\{0\} = V^0 \subset V^1 \subset \cdots \subset V^{k-1} \subset V^k = \mathcal{E} \tag{1.3}$$

and a positive su-β-Hölder function $\psi : \mathcal{M} \to \mathbb{R}$ so that the quotient-cocycles induced by the cocycle $\psi\mathcal{A}$ on V^i/V^{i-1} are uniformly bounded for $i = 1, ..., k$.

Let $\mathcal{B}$ be an su-β-Hölder fiber bunched cocycle over f on $\mathcal{E}'$. Then any μ-measurable conjugacy between $\mathcal{A}$ and $\mathcal{B}$ coincides μ-a.e. with an su-β-Hölder conjugacy which intertwines their holonomies (see Definition 2.4).

This theorem extends both the partially hyperbolic result [KS16, Theorem 4.2] for uniformly quasiconformal $\mathcal{A}$ and the hyperbolic result [KSW23, Theorem 2.1].

In the hyperbolic case, DeWitt recently showed in [DW] that fiber bunching of $\mathcal{B}$ can be verified if $\mathcal{B}$ is measurably conjugate to a cocycle $\mathcal{A}$ taking values in a Zimmer block. This assumption on $\mathcal{A}$ is weaker than in Theorem 1.4 and stronger than in Theorem 1.6. This result strongly relies on hyperbolicity and periodic points.

One of the difficulties in the partially hyperbolic case compared to the hyperbolic one is obtaining global regularity of conjugacies or invariant objects from (essential) regularity along the stable and unstable foliations. This step is simple for Hölder regularity in the hyperbolic case due to the local product structure of the stable and unstable foliations. To obtain continuity in the partially hyperbolic case we use results by Avila, Santamaria, and Viana [ASV13] for accessible center bunched volume preserving f. For *scalar* cocycles, global Hölder continuity of the conjugacy (with reduced Hölder exponent) was established by Wilkinson [W13]. However, accessibility is not known to yield global Hölder continuity of conjugacies or invariant objects for $GL(d,\mathbb{R})$-valued cocycles. This creates a mismatch between Hölder input and continuous output of the results, and hence difficulties in using them repeatedly or inductively, as continuity is not enough to work with. We overcome these difficulties by using holonomies and by obtaining the results with su-β-Hölder regularity for both input and output. We also give more general versions for various earlier results under the assumptions of su-β-Hölder regularity or existence of holonomies.

We also establish a result of independent interest on continuity of measurable solutions for twisted vector-valued cohomological equations over partially hyperbolic systems, Theorem 3.4, which covers the usual (untwisted) scalar and vector-valued cocycles as particular cases. This result plays a key role in the proof of Theorem 1.6.

The paper is structured as follows. We describe the setting and introduce the terminology in Section 2. We prove Theorem 1.6 in Section 4, and deduce Theorems 1.4 and 1.5 in Section 5. The results for twisted cohomological equations are stated and proved in Section 3, and those on regularity of invariant subbundles, Riemannian metrics, and conformal structures in Section 4.2.

2. Preliminaries

2.1. Partially hyperbolic diffeomorphisms.

Let $\mathcal{M}$ be a compact connected smooth manifold. A diffeomorphism f of $\mathcal{M}$ is *partially hyperbolic* if there exist a Df-invariant splitting of the tangent bundle

$$T\mathcal{M} = E^s \oplus E^c \oplus E^u$$

with non-trivial E^s and E^u, and a Riemannian metric on $\mathcal{M}$ for which one can choose continuous positive functions $\nu < 1 < \hat{\nu}$, γ, $\hat{\gamma}$ such that for any $x \in \mathcal{M}$ and unit vectors $\mathbf{v}^s \in E^s(x)$, $\mathbf{v}^c \in E^c(x)$, and $\mathbf{v}^u \in E^u(x)$

$$\|Df_x(\mathbf{v}^s)\| < \nu(x) < \gamma(x) < \|Df_x(\mathbf{v}^c)\| < \hat{\gamma}(x) < \hat{\nu}(x) < \|Df_x(\mathbf{v}^u)\|. \tag{2.1}$$

The sub-bundles E^s, E^u, and E^c are called, respectively, stable, unstable, and center. E^s and E^u are tangent to the stable and unstable foliations W^s and W^u respectively. We denote by $W^s_{\text{loc}}(x)$ the *local stable manifold*, which is the ball in $W^s(x)$ centered at x of a sufficiently small fixed radius, in the distance dist_{W^s} along the leaf.

An *su-path* in $\mathcal{M}$ is a concatenation of finitely many subpaths which lie entirely in a single leaf of W^s or W^u. A partially hyperbolic diffeomorphism f is called *accessible* if any two points in $\mathcal{M}$ can be connected by an *su*-path.

We say that f is *volume-preserving* if it has an invariant probability measure μ in the measure class of a volume induced by a Riemannian metric.

The diffeomorphism f is called *center bunched* if the functions $\nu, \hat{\nu}, \gamma, \hat{\gamma}$ can be chosen to satisfy $\gamma^{-1}\hat{\gamma} < \nu^{-1}$ and $\gamma^{-1}\hat{\gamma} < \hat{\nu}$. This implies that nonconformality of $Df|_{E^c}$ is dominated by contraction/expansion in E^s/E^u.

We recall that f is *hyperbolic* if $E^c = \mathbf{0}$. Hyperbolic diffeomorphisms are trivially center bunched, and accessible by the local product structure of stable and unstable manifolds. So our results apply to hyperbolic volume-preserving diffeomorphisms.

2.2. **Hölder continuous vector bundles.**

We consider a d-dimensional β-Hölder, $0 < \beta \le 1$, vector bundle $P : \mathcal{E} \to \mathcal{M}$. This means that there exists an open cover $\{U_i\}_{i=1}^k$ of $\mathcal{M}$ with coordinate systems

$$\phi_i : P^{-1}(U_i) \to U_i \times \mathbb{R}^d, \quad \phi_i(v) = (P(v), \Phi_i(v))$$

such that every transition map $\phi_j \circ \phi_i^{-1}$ is a homeomorphism and its restriction to the fiber $\Phi_j \circ \Phi_i^{-1}|_{\{x\}\times\mathbb{R}^d}$ depends β-Hölder continuously on x as a linear map on $\mathbb{R}^d$. We can identify $\mathcal{E}$ with a β-Hölder sub-bundle of a trivial bundle via

$$\phi : \mathcal{E} \to \mathcal{M} \times \mathbb{R}^{kd} \quad \text{with} \quad \phi(v) = (P(v), \rho_1\Phi_1(v) \times \dots \times \rho_k\Phi_k(v)),$$

where $\{\rho_i\}$ is a β-Hölder partition of unity for $\{U_i\}$.

Using this embedding we equip $\mathcal{E}$ with the induced β-Hölder Riemannian metric, i.e., a family of inner products on the fibers. We define an identification $I_{x,y}$ of fibers at nearby points x and y by $I_{x,y} = \Pi_y^{-1} \circ \Pi_x : \mathcal{E}_x \to \mathcal{E}_y$, where Π_x is the orthogonal projection in $\mathbb{R}^{kd}$ from $\mathcal{E}_x$ to the subspace which is the middle point of the unique shortest geodesic between $\mathcal{E}_x$ and $\mathcal{E}_y$ in the Grassmannian of d-dimensional subspaces of $\mathbb{R}^{kd}$. The identifications $\{I_{x,y}\}$ satisfy $I_{x,y} = I_{y,x}^{-1}$ and vary β-Hölder continuously on a neighborhood of the diagonal in $\mathcal{M} \times \mathcal{M}$.

2.3. **su-β-Hölder functions.**

We say that a function ψ on $\mathcal{M}$ with values in a metric space is *s-β-Hölder* if ψ is continuous on $\mathcal{M}$ and β-Hölder along the leaves of the stable foliation W^s, in the sense that there exists a constant K such that

$$d(\psi(x), \psi(y)) \le K \operatorname{dist}_{W^s}(x,y)^\beta \ \text{ for all } x \in \mathcal{M} \text{ and } y \in W^s_{\text{loc}}(x).$$

We define u-β-Hölder functions similarly and say a function is *su-β-Hölder* if it is s-β-Hölder and u-β-Hölder.

In the bundle setting, we similarly define the notion of an su-β-Hölder subbundle $\mathcal{E}'$ of a β-Hölder vector bundle $\mathcal{E}$ by using identifications $I_{x,y}$, or equivalently using an embedding $\mathcal{E}' \subset \mathcal{E} \subset \mathcal{M} \times \mathbb{R}^{kd}$ and thus viewing $\mathcal{E}'_x$ as the Grassmannian-valued function. Using the embedding we can also define local identifications $I'_{x,y}$ for $\mathcal{E}'$ as we did for $\mathcal{E}$. They are continuous on a neighborhood of the diagonal in $\mathcal{M} \times \mathcal{M}$ and β-Hölder along the leaves of W^s and W^u in the above sense. Then for objects on an su-β-Hölder subbundle we define the notion of being su-β-Hölder using these identifications. In particular, using an embedding we can obtain an su-β-Hölder Riemannian metric on $\mathcal{E}'$.

2.4. **Linear cocycles.**
Let f be a diffeomorphism of $\mathcal{M}$ and let $P : \mathcal{E} \to \mathcal{M}$ be a β-Hölder vector bundle over $\mathcal{M}$. A *linear cocycle* over f is an automorphism of $\mathcal{E}$ that projects to f, that is, a homeomorphism $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ such that $P \circ \mathcal{A} = f \circ P$ and for each $x \in \mathcal{M}$ the map $\mathcal{A}_x : \mathcal{E}_x \to \mathcal{E}_{fx}$ between the fibers is a linear isomorphism. In the case of a trivial vector bundle $\mathcal{E} = \mathcal{M} \times \mathbb{R}^d$, any linear cocycle $\mathcal{A}$ can be identified with a $GL(d, \mathbb{R})$-valued cocycle generated by the function $A(x) = \mathcal{A}_x \in GL(d, \mathbb{R})$.

We use the following notations for the iterates of $\mathcal{A}$: $\mathcal{A}_x^0 = \mathrm{Id}$, and for $n \in \mathbb{N}$,

$$\mathcal{A}_x^n = \mathcal{A}_{f^{n-1}x} \circ \cdots \circ \mathcal{A}_{fx} \circ \mathcal{A}_x : \mathcal{E}_x \to \mathcal{E}_{f^n x} \quad \text{and}$$
$$\mathcal{A}_x^{-n} = (\mathcal{A}_{f^{-n}x}^n)^{-1} : \mathcal{E}_x \to \mathcal{E}_{f^{-n}x}.$$

The prime examples of linear cocycles over f are the differential Df viewed as an automorphism of the tangent bundle $T\mathcal{M}$, and its restrictions to Df-invariant subbundles $\mathcal{E}' \subset T\mathcal{M}$ such as E^s, E^u, or E^c. In these examples,

$$\mathcal{A}_x = D_x f \;\text{ and }\; \mathcal{A}_x^n = D_x f^n, \quad \text{or} \quad \mathcal{A}_x = Df|_{\mathcal{E}'(x)} \text{ and } \mathcal{A}_x^n = Df^n|_{\mathcal{E}'(x)}.$$

Since these sub-bundles are Hölder continuous but usually not more regular, the Hölder category is natural for applications.

A linear cocycle $\mathcal{A}$ is called *β-Hölder* if $\mathcal{A}_x$ depends β-Hölder continuously on x, more precisely, if there exist a constant c such that for all nearby points $x, y \in \mathcal{M}$

$$\|\mathcal{A}_x - I_{fx,fy}^{-1} \circ \mathcal{A}_y \circ I_{x,y}\| \le c \cdot \mathrm{dist}(x,y)^\beta \tag{2.2}$$

where $\|.\|$ is the operator norm. Similarly, we say that $\mathcal{A}$ is *su-β-Hölder* if it is continuous and satisfies (2.2) for points x and y in the same local stable and unstable leaves. This notion is also defined in the same way for a cocycle $\mathcal{A}$ on an su-β-Hölder subbundle $\mathcal{E}'$ of $\mathcal{E}$ by using local identifications $I'_{x,y}$ on $\mathcal{E}'$.

Finally, we define the notion of conjugacy between linear cocycles.

Definition 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be linear cocycles over f on vector bundles $\mathcal{E}$ and $\mathcal{E}'$ over $\mathcal{M}$. Let $\mathcal{L} = \mathcal{L}(\mathcal{E}, \mathcal{E}')$ be the bundle whose fiber $\mathcal{L}_x$ is the space $L(\mathcal{E}_x, \mathcal{E}'_x)$ of linear operators from $\mathcal{E}_x$ to $\mathcal{E}'_x$. A (μ-measurable, continuous)* conjugacy *C between $\mathcal{A}$ and $\mathcal{B}$ is a (μ-measurable, continuous) section of $\mathcal{L}$ taking values in invertible operators and satisfying equation* (1.1).

2.5. **Holonomies and fiber bunching.** An important role in the study of cocycles, and in this paper in particular, is played by holonomies. Their existence was established for fiber bunched cocycles.

Definition 2.2. *An su-β-Hölder linear cocycle $\mathcal{A}$ is called* fiber bunched *if there exist constants $\theta < 1$ and K such that for all $x \in \mathcal{M}$ and $n \in \mathbb{N}$,*

$$\|\mathcal{A}_x^n\| \cdot \|(\mathcal{A}_x^n)^{-1}\| \cdot (\nu_x^n)^\beta < K\,\theta^n \quad \text{and} \quad \|\mathcal{A}_x^{-n}\| \cdot \|(\mathcal{A}_x^{-n})^{-1}\| \cdot (\hat{\nu}_x^{-n})^\beta < K\,\theta^n, \tag{2.3}$$

where ν and $\hat{\nu}$ are as in (2.1) *and*

$$\nu_x^n = \nu(f^{n-1}x) \cdots \nu(x) \quad \text{and} \quad \hat{\nu}_x^{-n} = (\hat{\nu}(f^{-n}x))^{-1} \cdots (\hat{\nu}(f^{-1}x))^{-1}. \tag{2.4}$$

Existence of holonomies was proved for $GL(d,\mathbb{R})$-valued cocycles in [AV10, ASV13] under a stronger fiber bunching assumption, and later extended to bundle setting in [KS13] and to the weaker fiber bunching (2.3) in [S15]. The proofs apply to su-β-Hölder cocycles without modifications.

Proposition 2.3. [AV10, ASV13, KS13, S15]
Let $\mathcal{A}$ be an su-β-Hölder linear cocycle over a partialy hyperbolic diffeomorphism $f : \mathcal{M} \to \mathcal{M}$. If $\mathcal{A}$ is fiber bunched, then for every $x \in \mathcal{M}$ and $y \in W^s(x)$ the limit

$$H_{x,y} = H_{x,y}^{\mathcal{A},s} = \lim_{n\to\infty} (\mathcal{A}_y^n)^{-1} \circ I_{f^n x, f^n y} \circ \mathcal{A}_x^n, \tag{2.5}$$

called a stable holonomy of $\mathcal{A}$, *exists and satisfies*

- (H1) *$H_{x,y}$ is an invertible linear map from $\mathcal{E}_x$ to $\mathcal{E}_y$;*
- (H2) *$H_{x,x} = Id$ and $H_{y,z} \circ H_{x,y} = H_{x,z}$, which implies $(H_{x,y})^{-1} = H_{y,x}$;*
- (H3) *$H_{x,y} = (\mathcal{A}_y^n)^{-1} \circ H_{f^n x, f^n y} \circ \mathcal{A}_x^n$ for all $n \in \mathbb{N}$;*
- (H4) *$\|H_{x,y} - I_{x,y}\| \le c\, dist(x,y)^\beta$, where c is independent of x and $y \in W^s_{loc}(x)$;*
- (H5) *The map $H^{\mathcal{A},s}: (x,y) \mapsto H_{x,y}^{\mathcal{A},s}$, where $x \in \mathcal{M}$ and $y \in W^s_{loc}(x)$, is continuous.*

The unstable holonomy

$$H_{x,y}^{\mathcal{A},u} = \lim_{n\to\infty} (\mathcal{A}_y^{-n})^{-1} \circ I_{f^{-n}x, f^{-n}y} \circ \mathcal{A}_x^{-n} \quad \text{for } y \in W^u(x)$$

also exists and satisfies similar properties.

By [KS13, Proposition 4.2], for a fiber bunched su-β-Hölder cocycle the map H satisfying (H1)-(H5) is unique. It follows that the stable and unstable holonomies do not depend on a particular choice of β-Hölder local identifications.

Definition 2.4. *A conjugacy C between $\mathcal{A}$ and $\mathcal{B}$* intertwines their holonomies *if*

$$H_{x,y}^{\mathcal{B},s/u} = C(y) \circ H_{x,y}^{\mathcal{A},s/u} \circ C(x)^{-1} \quad \text{for all } x,y \in \mathcal{M} \text{ with } y \in W^{s/u}(x). \tag{2.6}$$

3. Twisted cohomological equation

In this section, f is as in Assumptions 1.3, $\mathcal{E}$ is a β-Hölder vector bundle over $\mathcal{M}$, or more generally an su-β-Hölder subbundle of a β-Hölder vector bundle over $\mathcal{M}$, and $\mathcal{F}$ is an su-β-Hölder linear cocycle on $\mathcal{E}$ over f. We study the cohomological equation over f twisted by $\mathcal{F}$ for sections of $\mathcal{E}$. We will use the main result of this section, Theorem 3.4, in the inductive process in the proof of Theorem 1.6.

We say that a section $\varphi : \mathcal{M} \to \mathcal{E}$ is an *$\mathcal{F}$-twisted coboundary over f* if there exists a section $\eta : \mathcal{M} \to \mathcal{E}$ satisfying the following twisted cohomological equation

$$\varphi(x) = \eta(x) - (\mathcal{F}_x)^{-1}(\eta(fx)) \quad \text{equivalently} \quad \eta(x) = \varphi(x) + (\mathcal{F}_x)^{-1}(\eta(fx)). \tag{3.1}$$

In Theorem 3.4 we will establish regularity of a measurable solution of (3.1) with uniformly bounded twist $\mathcal{F}$, and show its invariance under twisted holonomies, which we introduce below.

In the case of the trivial bundle $\mathcal{E} = \mathcal{M} \times \mathbb{R}^d$ and the trivial twist $\mathcal{F}_x = \mathrm{Id}$, (3.1) is the usual vector-valued cohomological equation $\varphi(x) = \eta(x) - \eta(fx)$. In particular, Theorem 3.4 generalizes the usual measurable Livsic theorem for scalar cocycles in the hyperbolic case and extends the corresponding partially hyperbolic result in [W13].

Definition 3.1. *We say that a linear cocycle $\mathcal{F} : \mathcal{E} \to \mathcal{E}$ is* dominated *if there exist numbers $\theta < 1$ and K such that for all $x \in \mathcal{M}$ and $n \in \mathbb{N}$,*

$$\|(\mathcal{F}_x^n)^{-1}\| \cdot (\nu_x^n)^\beta < K\,\theta^n \quad \text{and} \quad \|(\mathcal{F}_x^{-n})^{-1}\| \cdot (\hat{\nu}_x^{-n})^\beta < K\,\theta^n, \tag{3.2}$$

where ν_x^n and $\hat{\nu}_x^{-n}$ are as in (2.4). We say that $\mathcal{A}$ is uniformly bounded *if there exists K such that $\|\mathcal{F}_x^n\| \le K$ for all $x \in \mathcal{M}$ and $n \in \mathbb{Z}$.*

To study equation (3.1) we consider the following twisted trajectory sum for φ:

$$\Phi^n(x) = \varphi(x) + (\mathcal{F}_x)^{-1}(\varphi(fx)) + \cdots + (\mathcal{F}_x^{n-1})^{-1}(\varphi(f^{n-1}x)) \in \mathcal{E}_x. \tag{3.3}$$

Proposition 3.2. *Let $\varphi : \mathcal{M} \to \mathcal{E}$ be an su-β-Hölder section and let $\mathcal{F} : \mathcal{E} \to \mathcal{E}$ be an su-β-Hölder linear cocycle over f. Suppose that $\mathcal{F}$ is dominated and fiber bunched, and let $H_{y,x}^s = H_{y,x}^{\mathcal{F},s}$ be the stable holonomy for $\mathcal{F}$. Then the limit*

$$\Phi_{y,x}^s = \Phi_{x,y}^{\mathcal{F},\varphi,s} = \lim_{n\to\infty} (\Phi^n(x) - H_{y,x}^s \Phi^n(y))$$

exists for any $x \in \mathcal{M}$ and $y \in W^s(x)$ and satisfies

(Φ1) *$\Phi_{y,x}^s \in \mathcal{E}_x$;*

(Φ2) *$\Phi_{x,x}^s = 0$ and $\Phi_{z,x}^s = \Phi_{y,x}^s + H_{y,x}^s(\Phi_{z,y}^s)$;*

(Φ3) *$\|\Phi_{y,x}^s\| \le K' d(x,y)^\beta$ where K' is independent of $x \in \mathcal{M}$ and $y \in W_{loc}^s(x)$;*

(Φ4) *The map $\Phi^s :\ (x,y) \mapsto \Phi_{y,x}^s$, where $x \in \mathcal{M}$ and $y \in W_{loc}^s(x)$, is continuous.*

A similar result holds for

$$\Phi_{y,x}^u = \lim_{n\to-\infty} (\Phi^n(x) - H_{y,x}^u \Phi^n(y)).$$

Proof. Using (3.3) we expand

$$\Phi^n(x) - H_{y,x}^s \Phi^n(y) = \sum_{k=0}^{n-1} [(\mathcal{F}_x^k)^{-1}(\varphi(f^k x)) - (H_{y,x}^s \circ (\mathcal{F}_y^k)^{-1} \circ H_{f^k x, f^k y}^s)(H_{f^k y, f^k x}^s \varphi(f^k y))].$$

Since $H_{y,x}^s \circ (\mathcal{F}_y^k)^{-1} \circ H_{f^k x, f^k y}^s = (\mathcal{F}_x^k)^{-1}$ by (H3), the k^{th} term in the sum equals

$$(\mathcal{F}_x^k)^{-1}(\varphi(f^k x)) - (\mathcal{F}_x^k)^{-1}(H_{f^k y, f^k x}^s \varphi(f^k y)) = (\mathcal{F}_x^k)^{-1}\,[\varphi(f^k x) - H_{f^k y, f^k x}^s \varphi(f^k y)].$$

For all $x \in \mathcal{M}$ and $y \in W^s_{loc}(x)$ we have $d(f^k x, f^k y) \le \nu_x^k \, d(x,y)$ for all $k \ge 0$. Since φ is su-β-Hölder we have

$$\|\varphi(f^k x) - I_{f^k y, f^k x}\, \varphi(f^k y)\| \le K_1 (\nu_x^k d(x,y))^\beta,$$

and since $H^s_{f^k y, f^k x}$ is β-Hölder close to $I_{f^k y, f^k x}$ by (H4), we conclude that

$$\|\varphi(f^k x) - H^s_{f^k y, f^k x}\, \varphi(f^k y)\| \le K_2 (\nu_x^k \, d(x,y))^\beta.$$

Now using the first inequality in (3.2) we estimate

$$\|(\mathcal{F}_x^k)^{-1}\,[\varphi(f^k x) - H^s_{f^k y, f^k x}\, \varphi(f^k y)]\,\| \;\le\; \|(\mathcal{F}_x^k)^{-1}\| \cdot \|\varphi(f^k x) - H^s_{f^k y, f^k x}\, \varphi(f^k y)\|$$
$$\le\; \|(\mathcal{F}_x^k)^{-1}\| \cdot K_2 (\nu_x^k \, d(x,y))^\beta \;\le\; K_2 L\, \theta^k d(x,y)^\beta \quad \text{with } \theta < 1.$$

We conclude that the series

$$\sum_{k=0}^{\infty} [\, (\mathcal{F}_x^k)^{-1}(\varphi(f^k x)) - H^s_{y,x} (\mathcal{F}_y^k)^{-1}(\varphi(f^k y))\,] \;=\; \lim_{n\to\infty} (\Phi^n(x) - H^s_{y,x}\Phi^n(y))$$

converges uniformly over all $x \in \mathcal{M}$ and $y \in W^s_{\text{loc}}(x)$. This yields existence of $\Phi^s_{y,x}$ and property (Φ4). Further, we can estimate

$$\|\Phi^n(x) - H^s_{y,x}\Phi^n(y)\| \le \sum_{k=0}^{n-1} K_2 L\, \theta^k d(x,y)^\beta \le K' d(x,y)^\beta,$$

so that the limit satisfies $\|\Phi^s_{x,y}\| \le K' d(x,y)^\beta$, which gives ($\Phi$3). Property ($\Phi$1) is trivial and ($\Phi$2) follows by taking the limit in

$$\Phi^n(x) - H^s_{z,x}\Phi^n(z) = (\Phi^n(x) - H^s_{y,x}\Phi^n(y)) + (H^s_{y,x}\Phi^n(y) - H^s_{z,x}\Phi^n(z)) =$$
$$= (\Phi^n(x) - H^s_{y,x}\Phi^n(y)) + H^s_{y,x}(\Phi^n(y) - H^s_{z,y}\Phi^n(z)),$$

where we use $H^s_{z,x} = H^s_{y,x} \circ H^s_{z,y}$. □

We now introduce twisted holonomies, which we then use to analyze regularity of solutions of the twisted cohomological equation (3.1). These are the maps

$$\mathcal{H}^s_{x,y} = \mathcal{H}^{\mathcal{F},\varphi,s}_{x,y} : \mathcal{E}_x \to \mathcal{E}_y \;\text{ for } y \in W^s(x).$$

Proposition 3.3. *Let $\varphi : \mathcal{M} \to \mathcal{E}$ be an su-β-Hölder section and let $\mathcal{F} : \mathcal{E} \to \mathcal{E}$ be an su-β-Hölder linear cocycle over f. Suppose that $\mathcal{F}$ is dominated and fiber bunched, and let $H^s_{x,y}$ and $\Phi^s_{x,y}$ be as in Proposition 3.2. Then the maps*

$$\mathcal{H}^s_{x,y}(v) = H^s_{x,y}(v) + \Phi^s_{x,y} = H^s_{x,y}(v) + \lim_{n\to\infty} (\Phi^n(y) - H^s_{x,y}\Phi^n(x)) \tag{3.4}$$

called stable twisted holonomies, *exist for any $x \in \mathcal{M}$ and $y \in W^s(x)$ and satisfy*

($\mathcal{H}$1) *$\mathcal{H}_{x,y}$ is an invertible affine map from $\mathcal{E}_x$ to $\mathcal{E}_y$;*

($\mathcal{H}$2) *$\mathcal{H}_{x,x} = Id$ and $\mathcal{H}_{y,z} \circ \mathcal{H}_{x,y} = \mathcal{H}_{x,z}$;*

($\mathcal{H}$3) *The map $\mathcal{H}^{\mathcal{F},\varphi,s} :\ (x,y) \mapsto \mathcal{H}^s_{x,\,y}$ is continuous in $x \in \mathcal{M}$ and $y \in W^s_{loc}(x)$.*

Proof. This follows directly from the previous proposition. For $(\mathcal{H}2)$ we use $(\Phi 2)$:

$$\mathcal{H}_{y,z} \circ \mathcal{H}_{x,y}(v) = \mathcal{H}_{y,z}(H^s_{x,y}(v) + \Phi^s_{x,y}) = (H_{y,z} \circ H^s_{x,y})(v) + H_{y,z}(\Phi^s_{x,y}) + \Phi^s_{y,z} =$$

$$= H^s_{x,z}(v) + \Phi^s_{x,z} = \mathcal{H}_{x,z}.$$

□

The unstable holonomy $\mathcal{H}^{\mathcal{F},\varphi,u}_{x,y}$ is defined similarly for $y \in W^u(x)$ and an analogous result holds. Now we formulate and prove the main theorem of this section.

Theorem 3.4. *Let (f, μ) be as in Assumption 1.3 and let $\mathcal{E}$ be an su-β-Hölder subbundle of a β-Hölder vector bundle over $\mathcal{M}$. Let $\mathcal{F} : \mathcal{E} \to \mathcal{E}$ be an su-β-Hölder uniformly bounded cocycle over f. Let $\varphi : \mathcal{M} \to \mathcal{E}$ be an su-β-Hölder section, and let $\eta : \mathcal{M} \to \mathcal{E}$ be a μ-measurable section satisfying*

$$\varphi(x) = \eta(x) - (\mathcal{F}_x)^{-1}(\eta(fx)) \quad \textit{for } \mu\textit{-a.e. } x.$$

Then, up to modification on a set of measure zero, η is su-β-Hölder and invariant under the twisted holonomies, that is,

$$\eta(y) = \mathcal{H}^{\mathcal{F},\varphi,s/u}_{x,y}\,\eta(x) \quad \textit{for all } x \in X \textit{ and } y \in W^{s/u}(x).$$

In the proof of this theorem we will use the following terminology and results from [ASV13] for a more general bundle setting. We formulate them using our notations.

Definition 3.5. [ASV13, Definition 2.9] *Let $(\mathcal{M}, f)$ be a partially hyperbolic system, and let $\mathcal{N}$ be a continuous fiber bundle over $\mathcal{M}$. A* stable holonomy *on $\mathcal{N}$ is a family of β-Hölder homeomorphisms $h^s_{x,y} : \mathcal{N}_x \to \mathcal{N}_y$ with uniform $\beta > 0$, defined for all x, y in the same stable leaf of f and satisfying*

(a) *$h^s_{y,z} \circ h^s_{x,y} = h^s_{x,z}$ and $h^s_{x,x} = Id$,*
(b) *the map $(x, y, \eta) \mapsto h^s_{x,y}(\eta)$ is continuous when (x, y) varies in the set of pairs of points in the same local stable leaf.*

Unstable holonomy is defined similarly, for pairs of points in the same unstable leaf.

Definition 3.6. [ASV13, Definition 2.10] *A measurable section $\Psi : \mathcal{M} \to \mathcal{N}$ of the fiber bundle $\mathcal{N}$ is called s-invariant if $h^s_{x,y}(\Psi(x)) = \Psi(y)$ for every x, y in the same stable leaf and* essentially s-invariant *if this relation holds restricted to some full measure subset. The definition of u-invariance is analogous. Finally, Ψ is* bi-invariant *if it is both s-invariant and u-invariant, and it is* bi-essentially invariant *if it is both essentially s-invariant and essentially u-invariant.*

A set in $\mathcal{M}$ is called *bi-saturated* if it consists of full stable and unstable leaves. The term *refinable* for a topological space is introduced in [ASV13, Definition 2.11]. For us it suffices to note that, by the remark after this definition, every Hausdorff space with a countable basis of topology is refinable.

Theorem 3.7. [ASV13, Theorem D] *Let $f : \mathcal{M} \to \mathcal{M}$ be a C^2 partially hyperbolic center bunched diffeomorphism preserving a volume μ, and let $\mathcal{N}$ be a continuous fiber bundle with stable and unstable holonomies and with refinable fiber. Then,*

(a) *for every bi-essentially invariant section $\Psi : \mathcal{M} \to \mathcal{N}$, there exists a bi-saturated set $\mathcal{M}_\Psi$ with full measure, and a bi-invariant section $\tilde{\Psi} : \mathcal{M}_\Psi \to \mathcal{N}$ that coincides with Ψ at μ almost every point.*

(b) *if f is accessible then $\mathcal{M}_\Psi = \mathcal{M}$ and $\tilde{\Psi}$ is continuous.*

Now we prove Theorem 3.4.

Proof. Clearly, an su-β-Hölder uniformly bounded cocycle is both dominated and fiber-bunched. Hence $\mathcal{F}$ has holonomies, and Propositions 3.2 and 3.3 yield $\Phi^s_{x,y} = \Phi^{\mathcal{F},\varphi,s}_{x,y}$ and twisted holonomies $\mathcal{H}^s_{x,y} = \mathcal{H}^{\mathcal{F},\varphi,s}_{x,y}$. Iterating (3.1) we obtain

$$\begin{aligned}\eta(x) &= \varphi(x) + (\mathcal{F}_x)^{-1}(\eta(fx)) = \varphi(x) + (\mathcal{F}_x)^{-1}[\varphi(fx) + \mathcal{F}_{fx}(\eta(f^2x))] = \dots \\ &= \varphi(x) + (\mathcal{F}_x)^{-1}(\varphi(fx)) + \dots + (\mathcal{F}^{n-1}_x)^{-1}(\varphi(f^{n-1}x)) + (\mathcal{F}^n_x)^{-1}(\eta(f^nx)) \\ &= \Phi^n(x) + (\mathcal{F}^n_x)^{-1}(\eta(f^nx)).\end{aligned}$$

Let $y \in \mathcal{M}$ and $x \in W^s(y)$. Using the equation above for $\eta(x)$ and $\eta(y)$ we obtain

$$\eta(x) - H^s_{y,x}\,\eta(y) = \Phi^n(x) - H^s_{y,x}\Phi^n(y) + \Delta_n, \quad \text{where}$$

$$\Delta_n = (\mathcal{F}^n_x)^{-1}(\eta(f^nx)) - H^s_{y,x}(\mathcal{F}^n_y)^{-1}(\eta(f^ny)).$$

By Proposition 3.2, $(\Phi^n(x) - H^s_{y,x}\Phi^n(y))$ converges to $\Phi^s_{y,x}$.

Now we show that $\|\Delta_n\| \to 0$ along a subsequence for all x, y in a set of full measure. First we note that by property (H3) we have $H^s_{y,x} \circ (\mathcal{F}^n_y)^{-1} = (\mathcal{F}^n_x)^{-1} \circ H^s_{f^ny,f^nx}$. Hence

$$\Delta_n = (\mathcal{F}^n_x)^{-1}\left(\eta(f^nx) - H^s_{f^ny,f^nx}(\eta(f^ny))\right) = (\mathcal{F}^n_x)^{-1}(\Delta'_n),$$

where $\Delta'_n = \eta(f^nx) - H^s_{f^ny,f^nx}(\eta(f^ny))$. By uniform boundedness of $\mathcal{F}$ we obtain

$$\|\Delta_n\| \le \|(\mathcal{F}^n_x)^{-1}\| \cdot \|\Delta'_n\| \le K\|\Delta'_n\|.$$

Since the section $\eta : \mathcal{M} \to E$ is μ-measurable, by Lusin's theorem there exists a compact set $S \subset \mathcal{M}$ with $\mu(S) > 1/2$ such that η is uniformly continuous and hence bounded on S. Let Y be the set of points in $\mathcal{M}$ for which the frequency of visiting S equals $\mu(S)$. By Birkhoff Ergodic Theorem, $\mu(Y) = 1$.

If $x, y \in Y$, there exists a subsequence $n_i \to \infty$ such that $f^{n_i}x, f^{n_i}y \in S$ for all i. Since $y \in W^s(x)$, $d(f^{n_i}x, f^{n_i}y) \to 0$ and hence $\|\Delta'_{n_i}\| \to 0$ by uniform continuity and boundedness of η on S and property (H4) of H^s. Thus $\|\Delta_{n_i}\| \to 0$ and we obtain

$$\eta(x) = H^s_{y,x}\,\eta(y) + \Phi^s_{y,x} = \mathcal{H}^s_{y,x}(\eta(y)) \quad \text{for all } x, y \in Y \text{ with } x \in W^s(y). \tag{3.5}$$

This means that η is essentially s-invariant in the sense of Definition 3.6 with $\mathcal{N} = \mathcal{E}$ and $h^s_{x,y} = \mathcal{H}^s_{x,y}$. We note that properties $(\mathcal{H}2)$ and $(\mathcal{H}3)$ of Proposition 3.3 yield Properties (a) and (b) of Definition 3.6. Also, $\mathcal{H}^s_{x,y}$ are invertible affine maps by $(\mathcal{H}1)$, and hence they are Lipschitz homeomorphisms.

A similar argument shows that η is essentially u-invariant. Thus section η is bi-essentially invariant. Since $\mathcal{N} = \mathcal{E}$ is refinable, Theorem 3.7 applies and yields that, up to modification on a set of measure zero, η is continuous on $\mathcal{M}$. Now by continuity it follows that (3.5) holds for all $x, y \in \mathcal{M}$ with $x \in W^s(y)$, that is, η is s-invariant. Further, since $\Phi^s_{x,y}$ and $H^s_{x,y}$ are β-Hölder on $W^s_{\mathrm{loc}}(x)$ by (Φ3) and (H4) respectively, (3.5) yields that

$$\|\eta(x) - I_{y,x}\eta(y)\| \le \|(H^s_{y,x} - I_{y,x})\eta(y)\| + \|\Phi^s_{y,x}\| \le K'\|\eta(y)\|\, d(x,y)^\beta$$

for all $x, y \in \mathcal{M}$ with $x \in W^s_{\mathrm{loc}}(y)$. Since η is continuous on $\mathcal{M}$, and hence bounded, this means that η is s-β-Hölder. A similar argument shows that η is u-β-Hölder. This completes the proof of Theorem 3.4. □

4. Proof of Theorem 1.6

4.1. Continuity of measurable conjugacy in uniformly quasiconformal case.

An important ingredient in the proof of Theorem 1.6 is the following result, which extends [KS16, Theorem 4.2]. We recall that a cocycle $\mathcal{A}$ is *uniformly quasiconformal* if $\|\mathcal{A}^n_x\| \cdot \|(\mathcal{A}^n_x)^{-1}\|$ is uniformly bounded in $x \in \mathcal{M}$ and $n \in \mathbb{Z}$.

Theorem 4.1. *Let (f, μ) be as in Assumption 1.3.*

(i) Continuous version. *Let $\mathcal{E}$ and $\mathcal{E}'$ be continuous vector bundles over $\mathcal{M}$, and let $\mathcal{A}$ and $\mathcal{B}$ be continuous linear cocycles over f on $\mathcal{E}$ and $\mathcal{E}'$ respectively. Suppose that $\mathcal{A}$ and $\mathcal{B}$ have stable and unstable holonomies satisfying (H1,2,3,5) of Proposition 2.3, and $\mathcal{A}$ is uniformly quasiconformal. Then any μ-measurable conjugacy between $\mathcal{A}$ and $\mathcal{B}$ coincides on a set of full measure with a continuous conjugacy which intertwines the holonomies of $\mathcal{A}$ and $\mathcal{B}$.*

(ii) su-Hölder version. *Let $\mathcal{E}$ and $\mathcal{E}'$ be su-β-Hölder subbundles of β-Hölder vector bundles over $\mathcal{M}$. Let $\mathcal{A}$ be a uniformly quasiconformal su-β-Hölder linear cocycle over f on $\mathcal{E}$. Let $\mathcal{B}$ be an su-β-Hölder fiber bunched linear cocycle over f on $\mathcal{E}'$ or, more generally, a continuous linear cocycle with holonomies as in Proposition 2.3. Then any μ-measurable conjugacy between $\mathcal{A}$ and $\mathcal{B}$ coincides on a set of full measure with an su-β-Hölder conjugacy.*

Proof. Recall that $\mathcal{L} = \mathcal{L}(\mathcal{E}, \mathcal{E}')$ is the vector bundle with fiber $\mathcal{L}_x = L(\mathcal{E}_x, \mathcal{E}'_x)$. Let C be a μ-measurable conjugacy between $\mathcal{A}$ and $\mathcal{B}$, that is, a μ-measurable section of $\mathcal{L}$ taking values in invertible linear operators and satisfying

$$\mathcal{B}_x = C(fx) \circ \mathcal{A}_x \circ C(x)^{-1} \text{ for } \mu \text{ almost every } x.$$

The main part of the proof is showing that C intertwines the stable holonomies of $\mathcal{A}$ and $\mathcal{B}$ on a set of full measure.

Since C is μ-measurable and the bundle $\mathcal{L}$ has countable basis of topology, by Lusin's theorem there exists a compact set $S \subset \mathcal{M}$ with $\mu(S) > 1/2$ such that C is uniformly continuous on S. Let Y be the set of points in $\mathcal{M}$ for which the frequency of visiting S equals $\mu(S)$. By Birkhoff ergodic theorem, $\mu(Y) = 1$.

Suppose that $x, y \in Y$ and $y \in W^s(x)$. Then

$$
\begin{aligned}
&(\mathcal{B}_y^n)^{-1} \circ I_{f^n x f^n y} \circ \mathcal{B}_x^n = \\
&= \left(C(f^n y) \circ \mathcal{A}_y^n \circ C(y)^{-1}\right)^{-1} \circ I_{f^n x, f^n y} \circ C(f^n x) \circ \mathcal{A}_x^n \circ C(x)^{-1} \\
&= C(y) \circ (\mathcal{A}_y^n)^{-1} \circ C(f^n y)^{-1} \circ I_{f^n x, f^n y} \circ C(f^n x) \circ \mathcal{A}_x^n \circ C(x)^{-1} \\
&= C(y) \circ (\mathcal{A}_y^n)^{-1} \circ (I_{f^n x, f^n y} + \Delta_n) \circ \mathcal{A}_x^n \circ C(x)^{-1} \\
&= C(y) \circ (\mathcal{A}_y^n)^{-1} \circ I_{f^n x, f^n y} \circ \mathcal{A}_x^n \circ C(x)^{-1} + C(y) \circ (\mathcal{A}_y^n)^{-1} \circ \Delta_n \circ \mathcal{A}_x^n \circ C(x)^{-1}.
\end{aligned} \tag{4.1}
$$

We will show that the last term tends to 0 along a subsequence $\{n_i\}$ such that $f^{n_i}x, f^{n_i}y \in S$ for all i. Since $x, y \in Y$, such a subsequence exists by the choice of Y. First we note that for the map

$$\Delta_n = C(f^n y)^{-1} \circ I_{f^n x, f^n y} \circ C(f^n x) - I_{f^n x, f^n y} : \ \mathcal{E}_{f^n x} \to \mathcal{E}_{f^n y}$$

we have $\|\Delta_{n_i}\| \to 0$ as $i \to \infty$ since $\operatorname{dist}(f^{n_i}x, f^{n_i}y) \to 0$ for $y \in W^s(x)$. This follows from uniform continuity of C on the compact set S.

Since the norms of stable holonomies are uniformly bounded over pairs of points in local stable leaves, and since $\mathcal{A}$ is uniformly quasiconformal, we obtain

$$
\begin{aligned}
&\|(\mathcal{A}_y^n)^{-1}\| \cdot \|\mathcal{A}_x^n\| \ \le \ \|(\mathcal{A}_y^n)^{-1}\| \cdot \|H^{\mathcal{A},s}_{f^n y, f^n x}\| \cdot \|\mathcal{A}_y^n\| \cdot \|H^{\mathcal{A},s}_{x,y}\| \ \le \\
&\|(\mathcal{A}_y^n)^{-1}\| \cdot K_1 \|\mathcal{A}_y^n\| \ \le \ K_1\, K_2 \quad \text{for all } x \in \mathcal{M} \text{ and } y \in W^s_{\text{loc}}(x).
\end{aligned}
$$

Now it follows that

$$\|C(y) \circ (\mathcal{A}_y^{n_i})^{-1} \circ \Delta_{n_i} \circ \mathcal{A}_x^{n_i} \circ C(x)^{-1}\| \le K_1\, K_2\, \|\Delta_{n_i}\| \to 0 \quad \text{ as } i \to \infty.$$

Passing to the limit in (4.1) along the sequence n_i we obtain that C intertwines the stable holonomies $H^{\mathcal{A},s}$ and $H^{\mathcal{B},s}$ on a set of full measure:

$$H^{\mathcal{B},s}_{x,y} = C(y) \circ H^{\mathcal{A},s}_{x,y} \circ C(x)^{-1} \quad \text{for all } x, y \in Y \text{ such that } y \in W^s(x). \tag{4.2}$$

or equivalently

$$C(y) = H^{\mathcal{A},s}_{x,y} \circ C(x) \circ (H^{\mathcal{B},s}_{x,y})^{-1} \quad \text{for all } x, y \in Y \text{ such that } y \in W^s(x). \tag{4.3}$$

Similarly, we obtain that C intertwines the unstable holonomies $H^{\mathcal{A},u}$ and $H^{\mathcal{B},u}$ on a set of full measure. Together these imply that C is a bi-essentially invariant section, in the sense of Definition 3.6, of the bundle $\mathcal{N} = \mathcal{L}$ with stable holonomy maps $h^s_{x,y} : \mathcal{L}_x \to \mathcal{L}_y$ defined as

$$h^s_{x,y}(C) = H^{\mathcal{A},s}_{x,y} \circ C \circ (H^{\mathcal{B},s}_{x,y})^{-1}$$

and similarly defined unstable holonomies. Properties (H2) and (H5) of Proposition 2.3 imply that these holonomies satisfy Properties (a) and (b) of Definition 3.5. Also, the maps $h^s_{x,y}$ are invertible linear and hence are Lipschitz homeomorphisms. Since the space $\mathcal{L}_x = L(\mathcal{E}_x, \mathcal{E}'_x)$ is Hausdorff with a countable basis of topology, it is refinable and thus Theorem 3.7 applies and yields that, up to modification on a set of measure zero, C is continuous on $\mathcal{M}$. Now (4.2) shows that C intertwines the holonomies of $\mathcal{A}$ and $\mathcal{B}$ everywhere on $\mathcal{M}$. This completes the proof of the first part of the theorem.

In the second part, since $\mathcal{A}$ is su-β-Hölder, uniform quasiconformality gives fiber bunching, and hence existence of holonomies by Proposition 2.3. The cocycle $\mathcal{B}$ also has holonomies by Proposition 2.3 or by the assumption. Thus the first part applies and yields that the conjugacy C is continuous and intertwines the holonomies of $\mathcal{A}$ and $\mathcal{B}$. The latter means that (4.3) holds everywhere, and it follows that C is s-β-Hölder. Indeed, Proposition 2.3 (H4) gives β-Hölder continuity of $H^{\mathcal{A},s}$ and $H^{\mathcal{B},s}$ along W^s, which yields that of C. Similarly, C is also u-β-Hölder and thus su-β-Hölder. ☐

4.2. **Regularity results for measurable invariant structures.**
In this section we give more general versions of earlier results on regularity of measurable invariant subbundles, Riemannian metrics, and conformal structures for linear cocycles. We will use these results in the proof of Theorem 1.6

We denote by $\lambda_+(\mathcal{A},\mu)$ and $\lambda_-(\mathcal{A},\mu)$ the largest and smallest Lyapunov exponents of a linear cocycle $\mathcal{A}$ with respect to μ, given by the Oseledets Multiplicative Ergodic Theorem. For μ almost all $x \in \mathcal{M}$, they equal to the following limits

$$\lambda_+(\mathcal{A},\mu) = \lim_{n\to\infty} n^{-1} \ln \|\mathcal{A}_x^n\| \quad \text{and} \quad \lambda_-(\mathcal{A},\mu) = \lim_{n\to\infty} n^{-1} \ln \|(\mathcal{A}_x^n)^{-1}\|^{-1}. \tag{4.4}$$

A cocycle $\mathcal{A}$ *has one exponent* with respect to μ if $\lambda_+(\mathcal{A},\mu) = \lambda_-(\mathcal{A},\mu)$. We note that a cocycle with more than one Lyapunov exponent may have measurable invariant sub-bundles which are not continuous. In particular, the Lyapunov sub-bundle for the negative Lyapunov exponent of cocycle $\mathcal{A}$ as in (1.2) is measurable but not continuous, see [S13, Example 2.9]. In contrast, for cocycles with one Lyapunov exponent we have

Theorem 4.2. *Let (f,μ) be as in Assumptions 1.3, let $\mathcal{E}$ be an su-β-Hölder subbundle of a β-Hölder vector bundle over $\mathcal{M}$, and let $\mathcal{B}$ be a fiber bunched su-β-Hölder linear cocycle over f on $\mathcal{E}$. If $\lambda_+(\mathcal{B},\mu) = \lambda_-(\mathcal{B},\mu)$ then any μ-measurable $\mathcal{B}$-invariant subbundle $\mathcal{E}'$ of $\mathcal{E}$ coincides μ-a.e. with an su-β-Hölder sub-bundle invariant under $\mathcal{B}$ and under its holonomies.*

This is essentially [KS13, Theorem 3.3] with β-Hölder assumption on $\mathcal{B}$ weakened to su-β-Hölder and continuity of $\mathcal{E}'$ improved to su-β-Hölder in the conclusion.

Proof. The proof of Theorem 3.3 in [KS13] goes through essentially without change as it relies on Theorem C in [ASV13] to show holonomy invariance and continuity of $\mathcal{E}'$. Theorem C in [ASV13] requires only $\lambda_+(\mathcal{B},\mu) = \lambda_-(\mathcal{B},\mu)$, continuity of $\mathcal{B}$, and existence of holonomies, for which it suffices to have $\mathcal{B}$ fiber bunched and su-β-Hölder. Then holonomy invariance of $\mathcal{E}'$ and Hölder property (H4) of holonomies along W^s and W^u yield that $\mathcal{E}'$ is su-β-Hölder. ☐

Theorem 4.3. *Let (f,μ) be as in Assumptions 1.3 and let $\mathcal{E}$ be an su-β-Hölder subbundle of a β-Hölder vector bundle over $\mathcal{M}$. Let $\mathcal{B}$ be either a fiber bunched* *su-β-Hölder linear cocycle over f on $\mathcal{E}$, or, more generally, a continuous linear cocycle with holonomies as in Proposition 2.3. Then any $\mathcal{B}$-invariant μ-measurable Riemannian metric (resp. conformal structure) on $\mathcal{E}$ coincides μ-a.e. with an su-β-Hölder Riemannian metric (resp. conformal structure) invariant under $\mathcal{B}$ and under its holonomies.*

We recall that the space $\mathcal{T}$ of inner products on $\mathbb{R}^d$ identifies with the space of real symmetric positive definite $d \times d$ matrices, which is isomorphic to $GL(d,\mathbb{R})/SO(d,\mathbb{R})$. The group $GL(d,\mathbb{R})$ acts transitively on $\mathcal{T}$ via $A[D] = A^T D\, A$, where $A \in GL(d,\mathbb{R})$ and $D \in \mathcal{T}$. The space $\mathcal{T}$ is a Riemannian symmetric space of non-positive curvature when equipped with a certain $GL(d,\mathbb{R})$-invariant metric [La, Ch. XII, Theorem 1.2]. A conformal structure on $\mathbb{R}^d$, $d \geq 2$, is a class of proportional inner products. The space of conformal structures on $\mathbb{R}^d$ can be similarly identified with $SL(d,\mathbb{R})/SO(d,\mathbb{R})$, which is also a Riemannian symmetric space of non-positive curvature with a $GL(d,\mathbb{R})$-invariant metric. A Riemannian metric (resp. conformal structure) on a vector bundle $\mathcal{E}$ is a section of the corresponding bundle whose fiber at x is the space of inner products (resp. conformal structures) on $\mathcal{E}_x$. See [KS10] for more details.

Proof of Theorem 4.3. For a fiber bunched β-Hölder cocycle, global continuity of an invariant μ-measurable conformal structure was established in [KS13, Theorem 3.1]. The main step, [KS13, Proposition 4.4], proves essential holonomy invariance of the conformal structure. Fiber bunching and β-Hölder continuity are used only to obtain holonomies, and thus they can be replaced by assuming existence of holonomies or by fiber bunching and the su-β-Hölder property, which imply it. The global continuity and holonomy invariance, together with the Hölder property (H4) of holonomies along W^s and W^u, yield that the conformal structure is su-β-Hölder. This completes the proof in the conformal structure case.

The proof for a Riemannian metric is almost identical, using the space of inner products in place of the space of conformal structures, which have the same properties for the purpose of the proof, described above. Alternatively, the result can be deduced by obtaining an invariant conformal structure using the previous case, and then using boundedness of the cocycle to find a proper normalization. □

Corollary 4.4. *Let (f,μ) be as in Assumptions 1.3 and let $\mathcal{E}$ be an su-β-Hölder subbundle of a β-Hölder vector bundle over $\mathcal{M}$. Suppose that $\mathcal{B}$ is either an su-β-Hölder linear cocycle or a continuous linear cocycle with holonomies as in Proposition 2.3. If $\mathcal{B}$ is uniformly bounded (resp. uniformly quasiconformal) then $\mathcal{B}$ preserves an su-β-Hölder invariant Riemannian metric (resp. conformal structure) on $\mathcal{E}$ invariant under the holonomies of $\mathcal{B}$.*

Proof. We note that for an su-β-Hölder cocycle both uniform boundedness and uniform quasiconformality imply fiber bunching and give existence of holonomies. Thus $\mathcal{B}$ has holonomies and by the previous theorem it suffices to obtain an invariant measurable Riemannian metric (resp. conformal structure) on $\mathcal{E}$. In the case of a conformal structure, [KS10, Proposition 2.4] shows that any uniformly quasiconformal continuous linear cocycle (over any diffeomorphism f) preserves a bounded Borel measurable conformal structure. The same result holds in the case of a Riemannian metric for uniformly bounded cocycles, and argument carries over without changes. □

4.3. **Proof of Theorem 1.6.**

We consider the invariant flag (1.3) for $\mathcal{A}$ assumed in the theorem,

$$\{0\} = V^0 \subset V^1 \subset \cdots \subset V^{k-1} \subset V^k = \mathcal{E}, \text{ and the quotient-bundles } \tilde{U}^i = V^i/V^{i-1}.$$

We fix a background su-β-Hölder Riemannian metric g on $\mathcal{E}$. Then for $i = 1, \dots, k$, the orthogonal complement of V^{i-1} in V^i is an su-β-Hölder subbundle of $\mathcal{E}$, which we denote by U^i. Then we have $V^i = U^1 \oplus \cdots \oplus U^i$, but in general only $U^1 = V^1$ is $\mathcal{A}$-invariant while U^i with $i > 1$ are not.

We use the splitting $\mathcal{E} = U^1 \oplus \cdots \oplus U^k$ to define a block triangular structure for $\mathcal{A}$. We denote by $P^j : \mathcal{E} \to U^j$ the projection to the U^j component in this splitting, and define the blocks $\mathcal{A}^{j,i} : U^i \to U^j$ as $\mathcal{A}^{j,i} = P^j \circ \mathcal{A}|_{U^i}$. The invariance of the flag implies that $\mathcal{A}^{j,i} = 0$ for $j > i$.

The projection $P^i|_{V^i}$ induces a continuous bundle isomorphism between U^i and the quotient $\tilde{U}^i$. Here we use continuous category for quotient bundles and structures on them, since the su-β-Hölder regularity was defined only for subbundles. However, using this isomorphism we identify the quotient cocycle $\tilde{\mathcal{A}}^{(i)}$ on $\tilde{U}^i$ with a linear cocycle $\mathcal{A}^{(i)}$ on the su-β-Hölder subbundle U^i. The cocycle $\mathcal{A}^{(i)}$ is also su-β-Hölder, as $\mathcal{A}^{(i)}_x$ coincides with the block $\mathcal{A}^{i,i}_x$. By continuity of the isomorphism, since $\psi\tilde{\mathcal{A}}^{(i)}$ is uniformly bounded by the assumption, so is the cocycle $\psi\mathcal{A}^{(i)}$. Now Corollary 4.4 yields that $\psi\mathcal{A}^{(i)}$ preserves an su-β-Hölder Riemannian metric σ_i on U^i. By isomorphism, the quotient $\psi\tilde{\mathcal{A}}^{(i)}$ also preserves a continuous Riemannian metric $\tilde{\sigma}_i$ on $\tilde{U}^i$.

Proposition 4.5. *Let $\mathcal{A}$ and k be as in Theorem 1.6. Then there exists a constant c such that for all $x \in \mathcal{M}$ and $0 \neq n \in \mathbb{Z}$,*

$$\|(\psi\mathcal{A})^n_x\| \le c|n|^{k-1} \quad \textit{and} \quad \|\mathcal{A}^n_x\| \cdot \|(\mathcal{A}^n_x)^{-1}\| \le c|n|^{2(k-1)}.$$

In particular, $\mathcal{A}$ has one Lyapunov exponent for each f-invariant ergodic measure ν, that is $\lambda_+(\mathcal{A}, \nu) = \lambda_-(\mathcal{A}, \nu)$.

Proof. This follows from the proof of [KS13, Theorem 3.10] which uses only the invariant flag (1.3) with continuous invariant Riemannian metrics on the quotients. The last statement follows from the second inequality since

$$\lambda_+(\mathcal{A}, \nu) - \lambda_-(\mathcal{A}, \nu) = \lim_{n\to\infty} n^{-1} \ln(\|\mathcal{A}^n_x\| \cdot \|(\mathcal{A}^n_x)^{-1}\|) \quad \text{for } \nu \text{ a.e. } x.$$ □

Since we have $\lambda_+(\mathcal{A}, \mu) = \lambda_-(\mathcal{A}, \mu)$ and $\mathcal{B}$ is μ-measurably conjugate to $\mathcal{A}$, we also have $\lambda_+(\mathcal{B}, \mu) = \lambda_-(\mathcal{B}, \mu)$. This follows from an easy lemma:

Lemma 4.6. [KSW23, Lemma 4.4] *Let μ be an ergodic f-invariant measure. If C is a μ-measurable conjugacy between cocycles $\mathcal{A}$ and $\mathcal{B}$, then for μ a.e. x and for each vector $0 \neq u \in \mathcal{E}_x$ the forward (resp. backward) Lyapunov exponent of u under $\mathcal{A}$ equals that of $C(x)u$ under $\mathcal{B}$.*

Now we construct the corresponding flag structure for $\mathcal{B}$. Denoting $\mathcal{V}^i_x = C(x)V^i_x$ we obtain the corresponding flag of measurable $\mathcal{B}$-invariant sub-bundles

$$\{0\} = \mathcal{V}^0 \subset \mathcal{V}^1 \subset \mathcal{V}^2 \subset \cdots \subset \mathcal{V}^k = \mathcal{E}'.$$

Since $\lambda_+(\mathcal{B},\mu) = \lambda_-(\mathcal{B},\mu)$, and since $\mathcal{B}$ is su-β-Hölder and fiber bunched by the assumption, Theorem 4.2 yields that this flag for $\mathcal{B}$ is su-β-Hölder.

Similarly to the case of $\mathcal{A}$, for each $i = 1, ..., k$, we define the corresponding objects for $\mathcal{B}$: the continuous quotient bundle $\tilde{\mathcal{U}}^i = \mathcal{V}^i/\mathcal{V}^{i-1}$ with the induced quotient cocycle $\tilde{\mathcal{B}}^{(i)}$; the orthogonal complement $\mathcal{U}^i$ of $\mathcal{V}^{i-1}$ in $\mathcal{V}^i$ with respect to an su-β-Hölder background Riemannian metric g' on $\mathcal{E}'$; the projection $\mathcal{P}^i : \mathcal{E} \to \mathcal{U}^i$ for the su-β-Hölder splitting $\mathcal{E}' = \mathcal{U}^1 \oplus \cdots \oplus \mathcal{U}^k$; su-$\beta$-Hölder blocks $\mathcal{B}^{j,i} = \mathcal{P}^j \circ \mathcal{B}|_{U^i}$ with triangular structure $\mathcal{B}^{j,i} = 0$ for $j > i$; and the su-β-Hölder cocycle $\mathcal{B}^{(i)}$ on $\mathcal{U}^i$ with $\mathcal{B}^{(i)}_x = \mathcal{B}^{i,i}_x$ that is continuously isomorphic to the quotient cocycle $\tilde{\mathcal{B}}^{(i)}$.

We note that C does not necessarily map U^i to $\mathcal{U}^i$ for $i > 1$. We denote the restriction of C to U^i by C^i and define the blocks by $C^{j,i} = \mathcal{P}^j \circ C^i : U^i \to \mathcal{U}^j$. Since $\mathcal{V}^i_x = C(x)V^i_x$, we have $C^i(U^i) \subset \mathcal{V}^i$ and thus $C^{j,i} = 0$ for $j > i$, so that C also has the block triangular structure.

First we show that the diagonal blocks $C^{i,i} : U^i \to \mathcal{U}^j$ are su-β-Hölder, for $i = 1, \dots, k$. For this we note that $C^{i,i}$ gives a measurable conjugacy between su-β-Hölder cocycles $\mathcal{A}^{(i)}$ on U^i and $\mathcal{B}^{(i)}$ on $\mathcal{U}^j$. Recall that $\mathcal{A}^{(i)}$ is conformal with respect to metric σ_i. The cocycle $\mathcal{B}^{(i)}$ has holonomies as in Proposition 2.3 induced, via the quotient, by the holonomies of the cocycle $\mathcal{B}$, which is su-β-Hölder and fiber bunched by the assumption. Now part (ii) of Theorem 4.1 shows that $C^{i,i}$ is su-β-Hölder. Also, pushing the metric σ_i by $C^{i,i}$ to $\mathcal{U}^j$ we obtain a Riemannian metric τ_i for which $\mathcal{B}^{(i)}$ is conformal and $\psi\mathcal{B}^{(i)}$ is isometric.

We will now show inductively that the restriction of C to V^i is su-β-Hölder for $i = 1, \dots, k$. The base case $i = 1$ follows from the previous paragraph since $C|_{V^1} = C^{1,1}$.

Now we describe the inductive step. Assuming that the restriction of C to V^{i-1} is su-β-Hölder we show that so is the restriction to V^i. Since $V^i = U^i \oplus V^{i-1}$, it suffices to show that the restriction C^i of C to U^i is also su-β-Hölder. We establish this for its components $C^{j,i}$, $j = i, \dots, 1$, by induction. In the base case $j = i$ we already know that the diagonal block $C^{i,i}$ is su-β-Hölder.

Now we show that $C^{i-\ell,i}$, with $0 < \ell < i$, is su-β Hölder assuming that $C^{i-j,i}$ is su-β-Hölder for $j = 0, 1, \dots \ell - 1$. Using the conjugacy equation

$$\mathcal{B}_x \circ C_x = C_{fx} \circ \mathcal{A}_x$$

and equating the $(i-\ell,i)$ components we obtain

$$\mathcal{B}_x^{i-\ell,i-\ell}\circ C_x^{i-\ell,i}+\mathcal{B}_x^{i-\ell,i-\ell+1}\circ C_x^{i-\ell+1,i}+\dots+\mathcal{B}_x^{i-\ell,i}\circ C_x^{i,i}$$
$$= C_{fx}^{i-\ell,i-\ell}\circ\mathcal{A}_x^{i-\ell,i}+C_{fx}^{i-\ell,i-\ell+1}\circ\mathcal{A}_x^{i-\ell+1,i}+\dots+C_{fx}^{i-\ell,i}\circ\mathcal{A}_x^{i,i}$$

and hence

$$C_x^{i-\ell,i}=(\mathcal{B}_x^{i-\ell,i-\ell})^{-1}\circ C_{fx}^{i-\ell,i}\circ\mathcal{A}_x^{i,i}\,+D_x \tag{4.5}$$

where

$$D_x=(\mathcal{B}_x^{i-\ell,i-\ell})^{-1}\circ(C_{fx}^{i-\ell,i-\ell}\circ\mathcal{A}_x^{i-\ell,i}+\dots+C_{fx}^{i-\ell,i-1}\circ\mathcal{A}_x^{i-1,i})$$
$$-(\mathcal{B}_x^{i-\ell,i-\ell})^{-1}\circ(\mathcal{B}_x^{i-\ell,i-\ell+1}\circ C_x^{i-\ell+1,i}+\dots+\mathcal{B}_x^{i-\ell,i}\circ C_x^{i,i}).$$

Then equation (4.5) is of the form (3.1) with

$$\varphi_x=D_x,\;\;\eta_x=C_x^{i-\ell,i},\;\;\text{and}\;\;\mathcal{F}_x(\eta_x)=\mathcal{B}_x^{i-\ell,i-\ell}\circ\eta_x\circ(\mathcal{A}_x^{i,i})^{-1},$$

where we view $C_x^{i-\ell,i}$ and D_x as sections of the bundle $\mathcal{L}(U^i,\mathcal{U}^{i-\ell})$ whose fiber at x is the space $L(U_x^i,\mathcal{U}_x^{i-\ell})$ of linear maps from U_x^i to $\mathcal{U}_x^{i-\ell}$. This is a subbundle of the β-Hölder bundle $\mathcal{L}=\mathcal{L}(\mathcal{E},\mathcal{E}')$, where we view $L(U_x^i,\mathcal{U}_x^{i-\ell})$ as the subspace of those operators in $L(\mathcal{E}_x,\mathcal{E}'_x)$ for which all other blocks, with respect to the splittings $\mathcal{E}_x=\oplus U_x^i$ and $\mathcal{E}'_x=\oplus\mathcal{U}_x^i$, are zeros. Since the splittings are su-β-Hölder, so is the subbundle $\mathcal{L}(U^i,\mathcal{U}^{i-\ell})$. We also have that D_x is su-β-Hölder since we inductively know that all its terms are su-β-Hölder. Indeed, for the second term this follows from the assumption that $C^{i-j,i}$ is su-β-Hölder for $j=0,1,\dots\ell-1$, and for the first term this follows from the assumption that the restriction of C to V^{i-1} is su-β-Hölder and hence so are all blocks $C^{i-\ell,m}$ with $m\le i-1$. The linear cocycle $\mathcal{F}$ over f on the bundle $\mathcal{L}(U^i,\mathcal{U}^{i-\ell})$ is su-β-Hölder since so are $\mathcal{B}^{i-\ell,i-\ell}$ and $\mathcal{A}^{i,i}$. Moreover, $\mathcal{F}$ is uniformly bounded since the cocycles $\psi\mathcal{B}^{i-\ell,i-\ell}$ and $\psi\mathcal{A}^{i,i}$ are isometric respect to σ_i and τ_i and hence

$$\|\mathcal{F}_x(\eta_{fx})\|\le\|\psi\mathcal{B}_x^{i-\ell,i-\ell}\|\cdot\|\eta_{fx}\|\cdot\|(\psi\mathcal{A}_x^{i,i})^{-1}\|=\|\eta_{fx}\|.$$

Thus we can apply Theorem 3.4 and conclude that $C^{i-\ell,i}$ is su-β-Hölder.

The argument above applies to $\ell=1,\dots,i-1$ and we conclude that all $C^{1,i},\dots,C^{i,i}$ are su-β-Hölder. We also recall that $C^{j,i}=0$ for $j>i$, and thus the restriction C^i of C to U^i is also su-β-Hölder. This proves that so is the restriction of C to V^i and completes the inductive step. We conclude that C is su-β-Hölder, completing the proof of Theorem 1.6.

5. Proofs of Theorems 1.4 and 1.5

5.1. **Proof of Theorem 1.4.** Let A be the matrix generating cocycle $\mathcal{A}$ in Theorem 1.4. The one exponent assumption means that all eigenvalues of A have the same modulus ρ. Then the real Jordan canonical form of matrix $\rho^{-1}A$ has block triangular structure with orthogonal bocks on the diagonal. This yields the corresponding flag of invariant subbundles for the cocycle $\mathcal{A}$ with properties as in Theorem 1.6. Hence Theorem 1.6 implies Theorem 1.4.

5.2. **Proof of Theorem 1.5.** Now we deduce Theorem 1.5 from Theorem 1.4. Let A be the matrix generating the cocycle $\mathcal{A}$ and let $\rho_1 < \cdots < \rho_\ell$ be the distinct moduli of its eigenvalues. We consider the corresponding invariant splitting

$$\mathbb{R}^d = E^1 \oplus \cdots \oplus E^\ell, \tag{5.1}$$

where E^i denotes the sum of the generalized eigenspaces of A corresponding to the eigenvalues of modulus ρ_i. This gives a splitting of the trivial bundle $\mathcal{E} = \mathcal{M} \times \mathbb{R}^d$ into $\mathcal{A}$-invariant constant subbundles E^i. For any $\epsilon > 0$ there is a suitable norm on $\mathbb{R}^d$ with resect to which we have

$$(\rho_i - \epsilon)^n \le \|\mathcal{A}^n u\| \le (\rho_i + \epsilon)^n \quad \text{for any unit vector } u \in E^i \text{ and } n \in \mathbb{Z}. \tag{5.2}$$

Let $B(x) = \mathcal{B}_x : \mathcal{M} \to GL(d, \mathbb{R})$ be the generator of the cocycle $\mathcal{B}$. If B is sufficiently C^0 close to A, then $\mathcal{E} = \mathcal{M} \times \mathbb{R}^d$ has a continuous $\mathcal{B}$-invariant splitting C^0 close to (5.1),

$$\mathbb{R}^d = \mathcal{E}_x^1 \oplus \cdots \oplus \mathcal{E}_x^\ell,$$

for which estimates similar to (5.2) hold,

$$(\rho_i - 2\epsilon)^n \le \|\mathcal{B}^n u\| \le (\rho_i + 2\epsilon)^n \quad \text{for any unit vector } u \in \mathcal{E}^i \text{ and } n \in \mathbb{Z}.$$

Moreover, for a Hölder $\mathcal{B}$ it is well known that the splitting is also Hölder with some exponent $\beta > 0$, which may be smaller than that of $\mathcal{B}$. See for example [KSW23, Lemma 5.1], which gives estimates for β in terms of ρ_i and f. We conclude that all restrictions $\mathcal{B}_i = \mathcal{B}|_{\mathcal{E}^i}$ are β-Hölder and hence are fiber bunched if ϵ is sufficiently small.

Let C be a measurable conjugacy between $\mathcal{A}$ and $\mathcal{B}$. We claim that C maps E^i to $\mathcal{E}^i$, that is, $C_x(E^i) = \mathcal{E}_x^i$ for μ a.e. x. Indeed, by Lemma 4.6, for μ a.e. x and for each unit vector $u \in E_x^i$ the forward and backward Lyapunov exponent of $C_x(u)$ is $\ln \rho_i$. This yields that $C_x(u) \in \mathcal{E}^i$, as having a non-zero component in another $\mathcal{E}^j$ would imply having forward or backward Lyapunov exponent under $\mathcal{B}$ different from $\ln \rho_i$ if ϵ is sufficiently small. Then $C_i = C|_{E^i}$ is a measurable conjugacy between the constant cocycle $\mathcal{A}_i = \mathcal{A}|_{E^i}$ with one Lyapunov exponent and the β-Hölder fiber bunched cocycle $\mathcal{B}_i$. By Theorem 1.4 each C_i, $i = 1, \dots, \ell$, is su-β-Hölder and hence so is C. This completes the proof of Theorem 1.5.

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