PERIODIC DATA RIGIDITY FOR COCYCLES AND HYPERBOLIC AUTOMORPHISMS

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Abstract. We study cohomology of Hölder continuous linear cocycles over a hyperbolic dynamical system and regularity of conjugacy between Anosov systems.

For cocycles $\mathcal{A}$ and $\mathcal{B}$ with conjugate periodic data, we establish Hölder cohomology under various conditions: the periodic data of $\mathcal{B}$ has narrow spectrum and the periodic data conjugacy $C(p)$ is Hölder continuous at a periodic point; $\mathcal{B}$ is constant and the cocycles are measurably cohomologous; $\mathcal{B}$ is constant and diagonalizable over $\mathbb{C}$ and either its Lyapunov spaces are at most two-dimensional or $C(p)$ is in a bounded set.

We also prove that a topological conjugacy between a weakly irreducible hyperbolic automorphism L and an Anosov diffeomorphism f of $\mathbb{T}^d$ is smooth if their derivative cocycles L and Df are conjugate. Using this and our results on cohomology of cocycles we obtain global periodic data rigidity results for weakly irreducible hyperbolic automorphisms. In the argument we also establish differentiability of stable holonomies in low regularity setting.

1. Introduction and results

1.1. **Introduction.** Cohomology of Hölder continuous cocycles over hyperbolic dynamical systems has been extensively studied. One of the main problems in this area is establishing cohomology based on the periodic data of the cocycles. It was solved by Livšic in [Li71, Li72] for scalar cocycles and, more generally, for cocycles with values in abelian groups. In the non-commutative setting, matrix-valued and linear cocycles are the primary objects of study, motivated by derivative cocycles of smooth systems. Cocycles play an important role in smooth dynamics and rigidity of Anosov systems. We formulate our results for linear cocycles in Section 1.2 and give applications to rigidity of hyperbolic automorphisms in Section 1.3.

Let $f : X \to X$ be a homeomorphism of a compact metric space and let $P : \mathcal{E} \to X$ be a finite dimensional Hölder continuous vector bundle over X. A *linear cocycle* $\mathcal{A}$ over f is an automorphism of $\mathcal{E}$ that projects to f, that is, a homeomorphism of $\mathcal{E}$ such that

$$P \circ \mathcal{A} = f \circ P \quad \text{and} \quad \mathcal{A}_x : \mathcal{E}_x \to \mathcal{E}_{fx} \text{ is a linear isomorphism for each } x \in X.$$

If f is a diffeomorphism of a manifold X, then the differential Df is a linear cocycle on the tangent bundle TX. Another important class of examples is given by random

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and Markovian sequences of matrices. They correspond to locally constant cocycles on a trivial bundle over a full shift or a subshift of finite type.

In the case of a trivial bundle $\mathcal{E} = X \times \mathbb{R}^d$, the cocycle is defined by a function $\mathcal{A} : X \to GL(d, \mathbb{R})$ with $\mathcal{A}(x) = \mathcal{A}_x$, and it is called a $GL(d, \mathbb{R})$ cocycle. We use the following metric on $GL(d, \mathbb{R})$

$$d(A, B) = \|A - B\| + \|A^{-1} - B^{-1}\|, \quad \text{where } \|\,.\,\| \text{ is the operator norm.}$$

A cocycle $\mathcal{A}$ is called β-Hölder if $\mathcal{A}_x$ depends β-Hölder continuously on x. A detailed description of this notion in the bundle setting is given in Section 2.2 of [KS13].

Two $GL(d, \mathbb{R})$ cocycles $\mathcal{A}$ and $\mathcal{B}$ over f are *cohomologous* if there exists a function $C : X \to GL(d, \mathbb{R})$ such that

$$\mathcal{A}_x = C(fx) \circ \mathcal{B}_x \circ C(x)^{-1} \quad \text{for all } x \in X.$$

Such a function C is called a *conjugacy* or *transfer map* between $\mathcal{A}$ and $\mathcal{B}$. For linear cocycles $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ and $\mathcal{B} : E \to E$ over f, a conjugacy is defined similarly with $C(x) \in GL(E_x, \mathcal{E}_x)$. In this case $C : E \to \mathcal{E}$ is a bundle isomorphism.

1.2. **Continuous conjugacy from the periodic data.**

Assumptions. In this section, $f : X \to X$ is a topologically transitive Anosov diffeomorphism, or a topologically mixing diffeomorphisms of a locally maximal hyperbolic set, or a mixing subshift of finite type (see Section 2 for details). All cocycles over f are assumed to be Hölder continuous.

We denote the nth iterate of $\mathcal{A}$ by $\mathcal{A}^n$, so that $\mathcal{A}^0 = \text{Id}$ and for $x \in X$ and $n \in \mathbb{N}$,

$$\mathcal{A}_x^n = \mathcal{A}_{f^{n-1}x} \circ \cdots \circ \mathcal{A}_x \quad \text{and} \quad \mathcal{A}_x^{-n} = (\mathcal{A}_{f^{-n}x}^n)^{-1}.$$

We say that $GL(d, \mathbb{R})$ cocycles $\mathcal{A}$ and $\mathcal{B}$ over f have *conjugate periodic data* if for each periodic point $p = f^n p$ there is a matrix $C(p) \in GL(d, \mathbb{R})$ such that

$$\mathcal{A}_p^n = C(p) \circ \mathcal{B}_p^n \circ C(p)^{-1}.$$

For linear cocycles this means existence of such an operator $C(p) \in GL(E_p, \mathcal{E}_p)$.

Clearly, continuous conjugacy between $\mathcal{A}$ and $\mathcal{B}$ implies conjugacy of their periodic data, and a natural question is whether the converse is true. This problem is difficult and far from fully solved. The following example shows that the answer may be negative even in dimension two with a constant cocycle $\mathcal{B}$ and a bounded set $\{C(p)\}$.

Example 1.1. [S13] *There exist a constant cocycle* $\mathcal{B}_x = \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}$ *and a smooth cocycle* $\mathcal{A}_x = \begin{bmatrix} 1 & \alpha(x) \\ 0 & 1 \end{bmatrix}$ *with* $\mathcal{A}_x$ *and* B_x *arbitrarily close to the identity, such that* $\mathcal{A}$ *and* $\mathcal{B}$ *have conjugate periodic data with* $\{C(p)\}$ *bounded in* $GL(2, \mathbb{R})$*, but are not measurably cohomologous.*

The case of $\mathcal{A}_p^n = \mathcal{B}_p^n$ is simpler and better understood. Positive answers were given by Parry and Pollicott [PaP97, Pa99] for cocycles with vales in compact groups and by Schmidt [Schm99] for cocycles with "bounded distortion". For general fiber bunched linear cocycles with equal periodic data the problem was solved in [S15, Bac15]. Positive results for conjugate periodic data were obtained in [Pa99, Schm99] for cohomology in compact groups under extra assumption of transitivity of the cocycle extension on the skew-product.

In light of Example 1.1, for linear cocycles one needs some regularity assumption on $C(p)$. We say that $C(p)$ is β-Hölder continuous at a periodic point q if

$$d(C(p), C(q)) \le c \operatorname{dist}(p,q)^\beta \quad \text{for every periodic point } p \text{ close to } q. \tag{1.1}$$

Under this assumption, Hölder cohomology between linear cocycles $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{B}$ is fiber-bunched (see Definition 3.2), was obtained in [S15, S17]. Further, Hölder cohomology was established for a constant $GL(d,\mathbb{R})$-valued cocycle $\mathcal{B}$ and a cocycle $\mathcal{A}$ sufficiently close to $\mathcal{B}$ with periodic data conjugacy satisfying (1.1). The following theorem extends this result to the global setting, and to $\mathcal{B}$ with δ-narrow periodic data, as defined in [DG24].

Definition 1.2. *A cocycle $\mathcal{B}$ has δ-*narrow periodic data *centered at $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_d$ if for each periodic point $p = f^n p$ the eigenvalues α_i of $\mathcal{B}_p^n$ can be ordered so that*

$$e^{n(\lambda_i - \delta)} \le |\alpha_i| \le e^{n(\lambda_i+\delta)} \quad \textit{for each } i = 1, \dots, d.$$

Clearly, this property holds for a constant cocycle with $\delta = 0$, and for a cocycle close to constant with a small $\delta > 0$.

Theorem 1.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be Hölder continuous linear cocycles over f on bundles $\mathcal{E}$ and E. Suppose that $\mathcal{A}$ and $\mathcal{B}$ have conjugate periodic data such that $C(p)$ can be chosen Hölder continuous at some periodic point q. Suppose also that $\mathcal{B}$ has δ-narrow periodic data with a sufficiently small δ. Then there exists a unique Hölder continuous conjugacy $\bar{C}$ between $\mathcal{A}$ and $\mathcal{B}$ with $\bar{C}(q) = C(q)$.*

Remark 1.4. *The Hölder exponent of $\bar{C}$ may be smaller than the exponents of $\mathcal{A}$ and $\mathcal{B}$, and it is discussed in the proof. The same holds in Theorems 1.5 and 1.6 below.*

We note that $\bar{C}(p)$ does not necessarily coincide with $C(p)$ for $p \ne q$. For example, let $\mathcal{B}_x = \mathrm{Id}$ and let $\mathcal{A}_x = \bar{C}(fx) \circ \bar{C}(x)^{-1}$, where $\bar{C} : X \to GL(d,\mathbb{R})$ is any Hölder continuous function with $\bar{C}(q) = \mathrm{Id}$. Then $\mathcal{A}_p^n = \mathcal{B}_p^n = \mathrm{Id}$ whenever $p = f^n p$, and thus we can take $C(p) = \mathrm{Id}$ for each p.

In the next theorem for a constant cocycle $\mathcal{B}$ we replace Hölder continuity of $C(p)$ at q by existence of a measurable conjugacy between the cocycles. We note that there are results on continuity of a measurable conjugacy, but not in this setting.

Theorem 1.5. *Let $\mathcal{B} = B$ be a constant $GL(d,\mathbb{R})$ cocycle and let $\mathcal{A}$ be a Hölder linear cocycle over f with conjugate periodic data. Suppose there exists a conjugacy C between $\mathcal{A}$ and $\mathcal{B}$ which is measurable with respect to an ergodic f-invariant measure on X with full support and local product structure. Then C coincides on a set of full measure with a Hölder continuous conjugacy between $\mathcal{A}$ and $\mathcal{B}$.*

In Theorem 1.5 and Theorem 1.6 below the linear cocycle $\mathcal{A}$ is on a vector bundle $\mathcal{E}$, and conjugacy of the cocycles yields that $\mathcal{E}$ is trivial.

In the next theorem we consider a constant cocycle $\mathcal{B}$ diagonalizable over $\mathbb{C}$. Compared to Theorem 1.3, we remove the continuity assumption on $C(p)$ if $\mathcal{B}$ has at most two-dimensional Lyapunov spaces and weaken it to boundedness in the general case.

Theorem 1.6. *Let $\mathcal{B} = B$ be a diagonalizable over $\mathbb{C}$ constant $GL(d, \mathbb{R})$ cocycle over f and let $\mathcal{A}$ be a Hölder linear cocycle over f with conjugate periodic data. Then $\mathcal{A}$ is Hölder conjugate to $\mathcal{B}$ if either of the following conditions holds*

- **(i)** *Lyapunov spaces of B are at most two-dimensional, that is, no three eigenvalues of B (counted with multiplicity) have the same modulus;*
- **(ii)** *The periodic data conjugacy $C(p)$ can be chosen in a bounded subset of $GL(d, \mathbb{R})$.*

This result is optimal for cohomology to a constant cocycle. Example 1.1 shows that the theorem does not hold without the diagonalizability assumption. Also, in dimension at least 3 the boundedness assumption on $C(p)$ cannot be dropped, as the following example by de la Llave in the appendix to [GKS11] shows.

Example 1.7. *Let f be an Anosov diffeomorphism of a manifold X. There exists a family of $SL(3, \mathbb{R})$-valued cocycles $\mathcal{A}_\epsilon$, $|\epsilon| < 1$, over f such that:*

- *$\mathcal{A}_\epsilon(x)$ is jointly analytic in ϵ and x;*
- *$\mathcal{A}_0 = B$ is a constant orthogonal matrix;*
- *For any ϵ and any $p = f^n p$, the matrix $\mathcal{A}_\epsilon^n(p)$ conjugate to $\mathcal{A}_0^n(p) = B^n$;*
- *For any $\epsilon \neq 0$, the cocycle $\mathcal{A}_\epsilon$ is not uniformly quasi-conformal, and so $\mathcal{A}_\epsilon$ cannot be continuously conjugate to $\mathcal{A}_0$.*

In this paper we use a recent result by DeWitt and Gogolev on existence of dominated splitting from periodic data [DG24]. Combing it with results and techniques developed in [S15, S17, KSW23] we obtain Theorems 1.3 and 1.5. Theorem 1.6 uses a new approach for removing/weakening continuity assumption on $C(p)$. The argument draws on results and ideas from [KS10, KS13, S15].

1.3. Applications to rigidity problems for hyperbolic systems.

We recall that a toral automorphism L is *hyperbolic* if it has no eigenvalues of modulus 1. By the classical results of Franks and Manning [F69, M73], any Anosov diffeomorphism f of $\mathbb{T}^d$ is topologically conjugate to the hyperbolic automorphism L that f induces on $\mathbb{Z}^d = H_1(\mathbb{T}^d, \mathbb{Z})$. A *topological conjugacy* is a homeomorphism h of $\mathbb{T}^d$ such that

$$L \circ h = h \circ f. \tag{1.2}$$

Any two such conjugacies differ by an affine automorphism of $\mathbb{T}^d$ commuting with L [Wa70], and hence have the same regularity. A conjugacy h in (1.2) is always bi-Hölder, but it is usually not even C^1, as there are various obstructions to smoothness.

The question when h is smooth has been extensively studied and periodic data played an important role [L87, LM88, L92, G08, GKS11, S15, DG24]. The problem is closely related to rigidity of cocycles. Indeed, if h is C^1 then differentiating (1.2) we obtain

$$L \circ Dh(x) = Dh(fx) \circ Df(x) \tag{1.3}$$

and hence $C(x) = Dh(x)$ is a continuous conjugacy of the derivative cocycles Df and $DL = L$. Thus conjugacy of the derivative cocycles is necessary for smoothness of h.

This condition is not sufficient in general for *reducible* L. This can be seen in the example by de la Llave [L92] of an automorphism $L(x, y) = (Ax, By)$ of $\mathbb{T}^4$ where $A, B \in SL(2, \mathbb{R})$ have eigenvalues λ, λ^{-1} and μ, μ^{-1}, respectively, with $\mu > \lambda > 1$. A perturbation

$$f(x, y) = (Ax + \varepsilon \sin(2\pi y_1)v,\ By),$$

where v is an eigenvector of A corresponding to λ, is not C^1 conjugate to L. For any $p = f^n p$, the matrix Df^n_p has eigenvalues $\lambda^n, \lambda^{-n}, \mu^n, \mu^{-n}$ and hence is conjugate to L^n, and Theorem 1.6(i) yields that cocycles Df and L are Hölder cohomologous.

Our main result in this section, Theorem 1.9, shows that conjugacy of the derivative cocycles is sufficient for *weakly irreducible* L. This assumption is weaker than irreducibility. We recall that $L \in SL(d, \mathbb{Z})$ is *irreducible* if its characteristic polynomial is irreducible over $\mathbb{Q}$, equivalently, if it has no nontrivial rational invariant subspaces. An automorphism L is irreducible if and only if every L-invariant linear foliation of $\mathbb{T}^d$ has dense leaves. The eigenvalues of an irreducible L are simple, and hence L is diagonalizable over $\mathbb{C}$, though different eigenvalues may have the same modulus. The notion of weak irreducibility was introduced in [KSW23, Section 3.3] in the context of bootstrapping the regularity of h to C^∞, and further discussed in [KS25, Section 2.2].

Definition 1.8. *We say that $L \in SL(d, \mathbb{Z})$ is* weakly irreducible *if any of the following equivalent conditions holds:*

(i) *The leaves of each Lyapunov foliation of L are dense in $\mathbb{T}^d$,*

(ii) *All irreducible over $\mathbb{Q}$ factors of the characteristic polynomial of L have the same set of moduli of the roots.*

A weakly irreducible L is not necessarily irreducible and not necessarily diagonalizable. For example, if A is weakly irreducible, then so are

$$L = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} A & I \\ 0 & A \end{pmatrix}.$$

Moreover, if $A, B \in SL(d, \mathbb{Z})$ are weakly irreducible with the same set of moduli of eigenvalues, then

$$L = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad \text{is weakly irreducible.}$$

Theorem 1.9. *Let $L : \mathbb{T}^d \to \mathbb{T}^d$ be a weakly irreducible hyperbolic automorphism, and let f be a $C^{1+\text{Hölder}}$ diffeomorphism of $\mathbb{T}^d$ topologically conjugate to L.*
Assume one of the following

(i) *The derivative cocycle Df is continuously conjugate to L,*

(ii) *The diffeomorphism f is sufficiently C^1 close to L, and there is a conjugacy between Df and L measurable with respect to an ergodic invariant measure (for f or for L) with full support and local product structure.*

Then f is $C^{1+Hölder}$ conjugate to L. Further, if f is C^∞ then it is C^∞ conjugate to L.

The C^∞ regularity is obtained using [KSW25]. The weak irreducibility assumption is dynamically natural, and examples of Gogolev in [G08] show that this assumption is necessary in the theorem. The proof of this theorem shares a general approach with [GKS11], however the proofs of the two main steps there, corresponding to Proposition 4.1 and verifying its assumption here, rely on conformality of L on its Lyapunov subspaces. In our setting possibility of Jordan blocks requires different arguments, and the proof of Proposition 4.1 uses a completely different approach. The only prior result for automorphisms with Jordan blocks was obtained by DeWitt [D25] in dimension 4.

In the proof of Theorem 1.9 we also establish a result of independent interest, Lemma 4.3, on $C^{1+\text{Hölder}}$ regularity of stable holonomies in $C^{1+\text{Hölder}}$ setting.

Combining Theorem 1.9(i) with our results for cocycles we obtain the following.

Corollary 1.10. *Let $L : \mathbb{T}^d \to \mathbb{T}^d$ be a weakly irreducible hyperbolic automorphism. Let f be a $C^{1+Hölder}$ Anosov diffeomorphism of $\mathbb{T}^d$ topologically conjugate to L with conjugate periodic data, that is, for each periodic point $p = f^n p$ there is*

$$C(p) \in GL(d, \mathbb{R}) \;\; \textit{such that} \;\; D_p f^n = C(p) \circ L^n \circ C(p)^{-1}.$$

Suppose that one of the following holds

- **(i)** *L is diagonalizable over $\mathbb{C}$ and no three of its eigenvalues have the same modulus,*
- **(ii)** *L is diagonalizable over $\mathbb{C}$ and $C(p)$ is in a compact subset of $GL(d, \mathbb{R})$,*
- **(iii)** *$C(p)$ is Hölder continuous at a periodic point of f, as in* (1.1).

Then f is $C^{1+Hölder}$ conjugate to L. Further, if f is C^∞ then any conjugacy is C^∞.

Hölder conjugacy of the derivative cocycles Df and $L = DL$ is given by Theorem 1.6 in (i) and (ii), and by Theorem 1.3 in (iii). Then Theorem 1.9(i) yields smooth conjugacy of f and L.

Corollary 1.10 is the most general periodic data rigidity result for Anosov toral automorphisms. Part (i) extends the similar global rigidity result in [DG24] for irreducible L to the weakly irreducible setting. Part (ii) gives an alternative to the assumption on the eigenvalues of L. Part (iii) generalizes a similar result in [S15] for perturbations of an irreducible L. Hölder assumption (1.1) yields smoothness of the conjugacy with minimal assumptions on L and f.

The diagonalizability assumption in (i) and (ii) is necessary. In [L02] de la Llave gave an example of an automorphism

$$L = \begin{pmatrix} A & I \\ 0 & A \end{pmatrix}, \qquad \text{where } A \in SL(2, \mathbb{Z}) \text{ is hyperbolic,}$$

and its analytic perturbation f with conjugate periodic data that is not C^1 conjugate to L. Clearly, L does not have three eigenvalues of the same modulus. One can also see that the $C(p)$ can be chosen bounded. For such automorphisms of $\mathbb{T}^4$, DeWitt [D25] identified an additional condition for the periodic data that ensures smooth conjugacy between f and L.

In Section 2 we describe the three main classes of hyperbolic systems. In Section 3 we prove our results for cocycles, Theorems 1.3, 1.5, and 1.6. In Section 4 we prove Theorem 1.9.

2. Hyperbolic systems in the base

We consider cocycles over hyperbolic dynamical systems. Below we describe the three main classes of such systems.

Transitive Anosov diffeomorphisms. A diffeomorphism f of a compact connected manifold X is called *Anosov* if there exist a splitting of the tangent bundle TX into a direct sum of two Df-invariant continuous subbundles $\mathcal{E}^s$ and $\mathcal{E}^u$, a Riemannian metric on X, and continuous functions ν and $\hat{\nu}$ such that

$$\|Df_x(v^s)\| < \nu(x) < 1 < \hat{\nu}(x) < \|Df_x(v^u)\| \tag{2.1}$$

for any $x \in X$ and any unit vectors $v^s \in \mathcal{E}^s(x)$ and $v^u \in \mathcal{E}^u(x)$. The sub-bundles $\mathcal{E}^s$ and $\mathcal{E}^u$ are called stable and unstable. They are tangent to the stable and unstable foliations $\mathcal{W}^s$ and $\mathcal{W}^u$ respectively. The *local stable manifold* of x, $\mathcal{W}^s_{\text{loc}}(x)$, is a ball centered at x of radius ρ in the intrinsic metric of $\mathcal{W}^s(x)$. We choose ρ sufficiently small so that $\mathcal{W}^s_{\text{loc}}(x) \cap \mathcal{W}^u_{\text{loc}}(z)$ consists of a single point for any sufficiently close x and z in X. Local unstable manifolds are defined similarly.

A diffeomorphism f is *(topologically) transitive* if there is a point x in X with dense orbit. All known examples of Anosov diffeomorphisms have this property. Any transitive Anosov diffeomorphism is *topological mixing*, that is, for any open sets $U_1, U_2 \subset X$ there exists $N \in \mathbb{N}$ such that $f^n(U_1) \cap U_2 \neq \emptyset$ for all $n \geq N$.

Topologically mixing diffeomorphisms of locally maximal hyperbolic sets. More generally, let f be a diffeomorphism of a manifold $\mathcal{M}$. A compact f-invariant set $X \subseteq \mathcal{M}$ is called *hyperbolic* if there exist a continuous Df-invariant splitting $T_X\mathcal{M} = E^s \oplus E^u$, and a Riemannian metric and continuous functions ν, $\hat{\nu}$ on an open set $U \supseteq X$ such that (2.1) holds for all $x \in X$. We assume that $f|X$ is topologically mixing.

The set X is called *locally maximal* if $X = \bigcap_{n\in\mathbb{Z}} f^{-n}(U)$ for some open set $U \supseteq X$.

Mixing subshifts of finite type. Let M be $k \times k$ matrix with entries from $\{0, 1\}$ such that all entries of M^N are positive for some $N \in \mathbb{N}$. Let

$$X = \{\, x = (x_n)_{n\in\mathbb{Z}} \,:\; 1 \leq x_n \leq k \;\text{ and }\; M_{x_n, x_{n+1}} = 1 \;\text{ for every } n \in \mathbb{Z} \,\}.$$

The shift map $f : X \to X$ is defined by $(fx)_n = x_{n+1}$. The system (X, f) is called a *mixing subshift of finite type*. We fix $\nu \in (0, 1)$ and consider the metric

$$\text{dist}(x, y) = d_\nu(x, y) = \nu^{n(x,y)}, \quad \text{where} \;\; n(x, y) = \min\{\, |i| \,:\, x_i \neq y_i \}.$$

The following sets play the role of the local stable and unstable manifolds of x:

$$\mathcal{W}^s_{\text{loc}}(x) = \{\, y \mid x_i = y_i, \;\; i \geq 0 \,\}, \quad \mathcal{W}^u_{\text{loc}}(x) = \{\, y \mid x_i = y_i, \;\; i \leq 0 \,\},$$

and we can take $\nu(x) = \nu$ and $\hat{\nu}(x) = \nu^{-1}$.

3. Proofs of Theorems 1.3, 1.5, and 1.6

3.1. Existence of a dominated splitting.

In the proofs, will be using the following result by DeWitt and Gogolev.

Theorem 3.1. [DG24, Theorems 1.3 and 3.10] *Let (X, f) be a transitive invertible subshift of finite type, or more generally a homeomorphism of a compact metric space satisfying the Closing Property* [DG24, Definition 3.9]. *Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$ with $\lambda_k > \lambda_{k+1}$. For each $\beta' \in (0,1)$ there exists $\delta > 0$ such that if $\mathcal{B}$ is a β'-Hölder linear cocycle with δ-narrow periodic data centered at $(\lambda_1, \ldots, \lambda_d)$ then $\mathcal{B}$ has a dominated splitting of index k.*

We will apply this result to mixing hyperbolic dynamical systems described in Section 2, for which the Closing Property holds.

A continuous splitting $E = G \oplus F$ is called a *dominated splitting* for a cocycle $\mathcal{B}$ on E if it is $\mathcal{B}$-invariant and there exist constants $K > 0$ and $0 < \tau < 1$ such that for every $x \in X$ and $n \in \mathbb{N}$,

$$\|\mathcal{B}_x^n | F_x\| < K\tau^n m(\mathcal{B}_x^n | G_x),$$

where $m(B) = \|B^{-1}\|^{-1}$ denotes the conorm of a linear operator B. We say that the dominated splitting has index k if $\dim G = k$.

3.2. Proof of Theorem 1.3.

We assume that $\mathcal{A}$ and $\mathcal{B}$ are Hölder continuous linear cocycles with some exponent $\beta' > 0$. We list the *distinct* values of $\lambda_1, \ldots, \lambda_d$ from the Definition 1.2 of δ-narrow periodic data for $\mathcal{B}$ as $\sigma_1 < \cdots < \sigma_\ell$ and denote by $d_1, \ldots, d_\ell$ their multiplicities. If $\delta > 0$ is sufficiently small, we can apply Theorem 3.1 to $\mathcal{B}$ and each gap $\sigma_i < \sigma_{i+1}$ and obtain the corresponding dominated splitting for $\mathcal{B}$ of the bundle E

$$E = E^1 \oplus \cdots \oplus E^\ell$$

into the direct sum of continuous invariant subbundles with $\dim E^i = d_i$ for $i = 1, \ldots, \ell$. Thus $\mathcal{B}$ splits into the direct sum of continuous cocycles $\mathcal{B}_i = \mathcal{B}|E^i$.

The subbundles E^i and hence cocycles $\mathcal{B}_i$ are Hölder continuous with some exponent $\hat{\beta} > 0$. Specifically, there are $\bar{\beta} = \bar{\beta}(f, \sigma_1, \ldots, \sigma_\ell)$ and $\delta_0 > 0$ such that for all $\delta < \delta_0$ we can take $\hat{\beta} = \min\{\beta', \bar{\beta}\}$. Let β'' be the exponent of Hölder continuity of $C(p)$ at q. We set $\beta = \min\{\beta'', \hat{\beta}\} > 0$. We will view $\mathcal{B}$ and $\mathcal{B}_i$ as β-Hölder cocycles and show the existence of β-Hölder conjugacy $\bar{C}$ between $\mathcal{A}$ and $\mathcal{B}$.

For each $\mathcal{B}_i$, the δ-narrow assumption implies that the Lyapunov exponents of $\mathcal{B}_i$ at each periodic orbit are in the interval $[\sigma_i - \delta, \sigma_i + \delta]$. Then by [K11, Theorem 1.3] for any $\delta' > \delta$ there is constant c such that

$$c^{-1} e^{n(\sigma_i - \delta')}\|u\| \leq \|\mathcal{B}_i^n u\| \leq c\, e^{n(\sigma_i + \delta')}\|u\| \quad \text{for any } i \text{ and any vector } u \in E^i.$$

In particular, if δ is small enough, then the β-Hölder cocycles $\mathcal{B}_i$ are β fiber bunched.

Definition 3.2. *A β-Hölder cocycle $\mathcal{B}$ over a hyperbolic system (X, f) is β* fiber bunched *if there exist numbers $\theta < 1$ and L such that for all $x \in X$ and $n \in \mathbb{N}$,*

$$\|\mathcal{B}_x^n\| \cdot \|(\mathcal{B}_x^n)^{-1}\| \cdot (\nu_x^n)^\beta < L\,\theta^n \quad \text{and} \quad \|\mathcal{B}_x^{-n}\| \cdot \|(\mathcal{B}_x^{-n})^{-1}\| \cdot (\hat{\nu}_x^{-n})^\beta < L\,\theta^n,$$

where ν and $\hat{\nu}$ are as in (2.1), *$\nu_x^n = \nu(f^{n-1}x) \cdots \nu(x)$ and $\hat{\nu}_x^{-n} = (\hat{\nu}_{f^{-n}x}^n)^{-1}$.*

Since $\mathcal{A}$ and $\mathcal{B}$ have conjugate periodic data, $\mathcal{A}$ also has δ-narrow periodic data centered at the exponents λ_i. Hence we also have the corresponding splittings for $\mathcal{A}$

$$\mathcal{E} = \mathcal{E}^1 \oplus \cdots \oplus \mathcal{E}^\ell \qquad \text{and} \qquad \mathcal{A} = \oplus \mathcal{A}_i,$$

where $\mathcal{A}_i = \mathcal{A}|\mathcal{E}^i$ satisfy the same estimates and are β-Hölder and fiber bunched.

At each periodic point the conjugacy $C(p)$ maps the splitting $E_x^1 \oplus \cdots \oplus E_x^\ell$ to the splitting $\mathcal{E}_x^1 \oplus \cdots \oplus \mathcal{E}_x^\ell$. Hence C induces the conjugacy $C_i(p)$ of the periodic data for $\mathcal{B}_i$ and $\mathcal{A}_i$, which is also β-Hölder continuous at q. Then by Theorem 1.9 in [S17], for each i there exists a unique β-Hölder conjugacy $\bar{C}_i$ between cocycles $\mathcal{B}_i$ and $\mathcal{A}_i$ with $\bar{C}_i(q) = C_i(q)$. Then $\bar{C} = \oplus \bar{C}_i$ is the claimed β-Hölder conjugacy between $\mathcal{A}$ and $\mathcal{B}$.

3.3. **Proof of Theorem 1.5.** As in the proof of Theorem 1.3, we obtain the splittings of $\mathcal{B}$ into the direct sum of constant cocycles $\mathcal{B}_i = \mathcal{B}|E^i$ and of $\mathcal{A}$ into the direct sum of Hölder continuous cocycles $\mathcal{A}_i = \mathcal{A}|\mathcal{E}^i$. Since each cocycle $\mathcal{B}_i$ is constant with one Lyapunov exponent λ_i, it is fiber bunched. The conjugacy of the periodic data implies that all Lyapunov exponents of $\mathcal{A}_i$ at the periodic orbits are also λ_i. By periodic approximation of Lyapunov exponents [K11, Theorems 1.4], this also holds for Lyapunov exponents of $\mathcal{A}_i$ for any f-invariant measure. Thus each cocycle $\mathcal{A}_i$ has one Lyapunov exponent λ_i with respect to every f-invariant measure.

Any measurable conjugacy preserves the Lyapunov exponents of vectors, see e.g. [KSW23, Lemma 4.4]. Hence the conjugacy C maps μ a.e. the splitting $E_x^1 \oplus \cdots \oplus E_x^\ell$ to the splitting $\mathcal{E}_x^1 \oplus \cdots \oplus \mathcal{E}_x^\ell$, and so it induces μ-measurable conjugacy C_i between $\mathcal{B}_i$ and $\mathcal{A}_i$. Since $\mathcal{B}_i$ is fiber bunched and $\mathcal{A}_i$ has one Lyapunov exponent, C_i coincides μ a.e. with a Hölder continuous conjugacy by [KSW23, Theorem 2.1]. This holds for each C_i, and hence we conclude the same for their direct sum C.

3.4. **Proof of Theorem 1.6.** Let $\mathcal{B}_x = B$ be the constant generator of the cocycle $\mathcal{B}$, which is diagonalizable over $\mathbb{C}$. Let $\rho_1 < \cdots < \rho_\ell$ be the distinct moduli of the eigenvalues of B and let $\mathbb{R}^d = E^1 \oplus \cdots \oplus E^\ell$ be the Lyapunov splitting for $\mathcal{B}$, that is the invariant splitting into the direct sums of the eigenspaces corresponding to eigenvalues of modulus ρ_i, $i = 1, \dots, \ell$. We denote $B_i = B|E^i$. Since $\mathcal{B}$ is diagonalizable over $\mathbb{C}$, there exists an inner product on $\mathbb{R}^d$ with respect to which

$$\|B_i^n u\| = \rho_i^n \|u\| \quad \text{for every } i = 1, \dots, \ell, \text{ every vector } u \in E^i \text{ and every } n.$$

In particular, the cocycle $\mathcal{B}_i$ generated by B_i is fiber bunched for any $\beta > 0$.

As in the proof of Theorem 1.3 we obtain the corresponding Hölder continuous invariant splitting $\mathcal{E} = \mathcal{E}^1 \oplus \cdots \oplus \mathcal{E}^\ell$ for $\mathcal{A}$ and split $\mathcal{A}$ into the sum of Hölder continuous cocycles $\mathcal{A}_i = \mathcal{A}|\mathcal{E}^i$. For any $p = f^n p$, the restriction of the conjugacy $C(p)$ to E^i

conjugates $(\mathcal{A}_i)_p^n$ and $(\mathcal{B}_i)_p^n$. Since all $(\mathcal{B}_i)_p^n$ are conformal, it follows from our results in [KS10] that the cocycle $\mathcal{A}_i$ is also conformal with respect to some Hölder continuous Riemannian metric on $\mathcal{E}^i$. Specifically, in 2-dimensional case (i) this follows from [KS10, Theorem 1.3] and in the general case (ii) under the boundedness assumption on $C(p)$ this follows from [KS10, Theorem 1.1].

With respect to this metric, the norm $a_i(x) = \|\mathcal{A}_i(x)\|$ is an $\mathbb{R}_+$-valued cocycle with periodic data equal to that of the constant cocycle ρ_i. Hence by the Livšic periodic point theorem, the cocycle a_i is Hölder continuously cohomologous to the constant cocycle ρ_i via a positive function $\phi(x)$. Then we can rescale the Riemannian metric on $\mathcal{E}^i$ by $\phi(x)$ and obtain that $\|\mathcal{A}_i(x)\| = \rho_i$ with respect to the adjusted metric. Thus we also have

$$\|\mathcal{A}_i^n v\| = \rho_i^n \|v\| \quad \text{for every vector } v \in \mathcal{E}^i \text{ and every } n.$$

We will show that the cocycles $\mathcal{A}_i$ and $\mathcal{B}_i$ are cohomologous by some Hölder continuous conjugacy $C_i(x)$. Then $C = \oplus C_i$ conjugates $\mathcal{A}$ and $\mathcal{B}$.

From now on we fix i. If $\dim(E^i) = 1$, then Hölder conjugacy of $\mathcal{A}_i$ and $\mathcal{B}_i$ is given by the Livšic periodic point theorem. So from now on we assume that $\dim(E^i) > 1$. We denote $\tilde{\mathcal{A}} = (\rho_i)^{-1}\mathcal{A}_i$ and $\tilde{B} = (\rho_i)^{-1}B_i$. Then $\tilde{B}$ is a constant isometric cocycle on the trivial bundle $\tilde{E} = E^i$ and $\tilde{\mathcal{A}}$ is a Hölder continuous cocycle on the Hölder continuous bundle $\tilde{\mathcal{E}} = \mathcal{E}^i$ isometric with respect to a Hölder continuous Riemannian metric σ on $\tilde{\mathcal{E}}$. The periodic data of these cocycles are conjugate by $\tilde{C}(p) = C_i(p) = C|E^i(p)$.

In case (ii) the set $\{\tilde{C}(p)\}$ is bounded by the assumption. We claim that in case (i) we also can choose $\{\tilde{C}(p)\}$ to be bounded. Indeed, we take any periodic point $p = f^n(p)$ and note that $\tilde{\mathcal{A}}_p^n$ is isometric with respect to an inner product $\sigma(p)$ on $\tilde{\mathcal{E}}$. Since σ is continuous and thus bounded, we can choose in a bounded way linear isomorphisms $D_p : \tilde{\mathcal{E}}_p \to \mathbb{R}^2$ which map $\sigma(p)$ to the standard inner product on $\mathbb{R}^2$. Then $D_p \circ \tilde{\mathcal{A}}_p^n \circ D_p^{-1}$ is an orthogonal matrix conjugate to the ortogonal matrix $\tilde{B}^n$ by $D_p\tilde{C}(p)$. Thus $D_p \circ \tilde{\mathcal{A}}_p^n \circ D_p^{-1}$ and $\tilde{B}^n$ are orthogonal matrices with the same eigenvalues and hence can be conjugate by an orthogonal matrix. It follows that $\tilde{\mathcal{A}}_p^n$ can be conjugate $\tilde{B}^n$ by a matrix in a bounded set.

First, we consider the case f when has a fixed point $q = f(q)$. Conjugating the cocycle $\mathcal{A}$ by $\tilde{C}(q)$ we can assume without loss of generality that the cocycles have the same value at q, that is $\tilde{\mathcal{A}}_q = \tilde{B}_q = \tilde{B}$, and that $\tilde{C}(q) = \mathrm{Id}$.

Since $\tilde{\mathcal{A}}$ is isometric, it is fiber bunched and hence has stable and unstable cocycle holonomies, that is,

$$H^s_{x,y} = H^{\tilde{\mathcal{A}},s}_{x,y} = \lim_{n\to\infty} (\tilde{\mathcal{A}}^n_y)^{-1} \circ \tilde{\mathcal{A}}^n_x \quad \text{exists for any } x \in X \text{ and any } y \in W^s(x), \text{ and}$$

$$H^u_{x,y} = H^{\tilde{\mathcal{A}},u}_{x,y} = \lim_{n\to\infty} ((\tilde{\mathcal{A}}^{-n}_y)^{-1} \circ \tilde{\mathcal{A}}^{-n}_x) \quad \text{exists for any } x \in X \text{ and any } y \in W^u(x),$$

Above, $W^s(x)$ and $W^u(x)$ denote the stable and unstable leaves of a point $x \in X$.

The holonomies for the constant cocycle $\tilde{B}$ are trivial, that is, equal to the identity. For a nontrivial Hölder continuous bundle $\tilde{\mathcal{E}}$ one needs to consider appropriate Hölder continuous identifications of fibers at nearby points in the formulas above, and we refer to [KS13] for more details. The existence and properties of holonomies (and their independence of the choice of identifications) for this setting were established in [KS13, Proposition 4.2]. The holonomies $H^{s/u}_{x,\,y} : \tilde{\mathcal{E}}_x \to \tilde{\mathcal{E}}_y$ are isomorphisms between the fibers of $\tilde{\mathcal{E}}$ satisfying a natural equivariance property

$$\tilde{\mathcal{A}}_x = H^{s/u}_{fy,\,fx} \circ \tilde{\mathcal{A}}_y \circ H^{s/u}_{x,\,y}. \tag{3.1}$$

They are also Hölder continuous along the leaves in the following sense

$$\|H^{s/u}_{x,y} - \mathrm{Id}\| \le K \mathrm{dist}(x,y)^\beta \quad \text{for all } x \in X \text{ and all } y \in W^{s/u}_{loc}(x), \tag{3.2}$$

where β is a Hölder exponent of the cocycle and the constant K depends on the size of the local leaves. This follows from the estimates [KS13, Proposition 4.2(i)]:

$$\begin{aligned} &\|(\tilde{\mathcal{A}}^n_y)^{-1} \circ \tilde{\mathcal{A}}^n_x - \mathrm{Id}\,\| \le K\,\mathrm{dist}(x,y)^\beta \quad \text{for all } x \in X,\ y \in W^s_{loc}(x),\ n \in \mathbb{N}, \\ &\|(\tilde{\mathcal{A}}^{-n}_y)^{-1} \circ \tilde{\mathcal{A}}^{-n}_x - \mathrm{Id}\,\| \le K\,\mathrm{dist}(x,y)^\beta \quad \text{for all } x \in X,\ y \in W^u_{loc}(x),\ n \in \mathbb{N}. \end{aligned} \tag{3.3}$$

First we define conjugacies C^s and C^u on the stable and unstable leaves of the fixed point q. We set

$$\begin{aligned} &C^s(q) = C^u(q) = \tilde{C}(q) = \mathrm{Id}, \\ &C^s(x) = H^s_{q,\,x} \quad \text{for } x \in W^s(q), \\ &C^u(x) = H^u_{q,\,x} \quad \text{for } x \in W^u(q). \end{aligned}$$

Since $q = f(q)$ and $\tilde{\mathcal{A}}_q = \tilde{B}$, using (3.1) we obtain

$$C^{s/u}(fx) \circ \tilde{B}_x \circ C^{s/u}(x)^{-1} = H^{s/u}_{q,\,fx} \circ \tilde{B} \circ (H^{s/u}_{q,\,x})^{-1} = H^{s/u}_{q,\,fx} \circ \tilde{\mathcal{A}}_q \circ H^{s/u}_{x,\,q} = \tilde{\mathcal{A}}_x.$$

So these are indeed conjugacies between $\tilde{\mathcal{A}}_x$ and $\tilde{B}_x$ on the corresponding leaves.

Now we show that if x is a homoclinic point for q, that is, $x \in S := W^s(q) \cap W^u(q)$, then

$$(C^s(x))^{-1} \circ C^u(x) = H^s_{x,q} \circ H^u_{q,x} = \mathrm{Id}, \quad \text{that is,} \;\; C^s(x) = C^u(x). \tag{3.4}$$

By compactness of the orthogonal group we can find a sequence $n_k \to \infty$ such that $\tilde{B}^{n_k} \to \mathrm{Id}$. Recall that we also have $\tilde{\mathcal{A}}^{n_k}_q = \tilde{B}^{n_k} \to \mathrm{Id}$. For this sequence the holonomies can be expressed as follows

$$H^s_{x,q} = \lim_n ((\tilde{\mathcal{A}}^n_q)^{-1} \tilde{\mathcal{A}}^n_x) = \lim_k ((\tilde{\mathcal{A}}^{n_k}_q)^{-1} \tilde{\mathcal{A}}^{n_k}_x) = \lim_k \tilde{\mathcal{A}}^{n_k}_x,$$

and similarly as $q = f^{-n_k}(q)$

$$H^u_{q,x} = \lim_k (\tilde{\mathcal{A}}^{-n_k}_x)^{-1} \tilde{\mathcal{A}}^{-n_k}_q = \lim_k \tilde{\mathcal{A}}^{n_k}_{f^{-n_k}(x)} (\tilde{\mathcal{A}}^{n_k}_{f^{-n_k}(q)})^{-1} = \lim_k \tilde{\mathcal{A}}^{n_k}_{f^{-n_k}(x)}.$$

We conclude that

$$(C^s(x))^{-1} \circ C^u(x) = H^s_{x,q} \circ H^u_{q,x} = \lim_k \tilde{\mathcal{A}}^{n_k}_x \circ \tilde{\mathcal{A}}^{n_k}_{f^{-n_k}(x)} = \lim_k \tilde{\mathcal{A}}^{2n_k}_{f^{-n_k}(x)}. \tag{3.5}$$

We consider the orbit segment $\{f^{-n_k}x, \dots, x, \dots f^{n_k}(x)\}$. Since both $f^{-n_k}x \to q$ and $f^{n_k}(x) \to q$ as $k \to \infty$, for large enough k

$$\delta_k = \operatorname{dist}(f^{-n_k}x, f^{n_k}(x))$$

is sufficiently small. Hence the Anosov Closing Lemma [KtH, 6.4.15-17]) for (X, f) applies and yields existence of a periodic point $p_k = f^{2n_k}(p_k)$ whose orbit is close to the orbit segment above, and in particular

$$\operatorname{dist}(f^{-n_k}x, p_k) \le K_1\delta_k \quad \text{and} \quad \operatorname{dist}((f^{n_k}x), p_k) \le K_1\delta_k.$$

Now we show that $\tilde{\mathcal{A}}^{2n_k}_{f^{-n_k}(x)}$ is close to $\tilde{\mathcal{A}}^{2n_k}_{p_k}$. By the local product structure there exits a unique point

$$y = W^s_{loc}(f^{-n_k}x) \cap W^u_{loc}(p_k).$$

Then $\operatorname{dist}(f^{-n_k}x, y) \le K_2\delta_k$, and the orbit segment $\{y, \dots, f^{2n_k}(x)\}$ is close to both orbit segments $\{f^{-n_k}x, \dots, f^{n_k}(x)\}$ and $\{p_k, \dots, f^{2n_k}(p_k) = p_k\}$. It follows that

$$f^{2n_k}y = W^s_{loc}(f^{n_k}x) \cap W^u_{loc}(p_k), \;\text{ and so }\; \operatorname{dist}(f^{2n_k}y, p_k) \le K_2\delta_k.$$

Then applying (3.3) to $f^{-n_k}x$ and y, and to $f^{2n_k}y$ and $f^{2n_k}p_k = p_k$ we obtain

$$\|(\tilde{\mathcal{A}}^{2n_k}_y)^{-1}\tilde{\mathcal{A}}^{2n_k}_{f^{-n_k}(x)} - \mathrm{Id}\| \le K(K_2\delta_k)^\beta \qquad \text{and}$$
$$\|\tilde{\mathcal{A}}^{2n_k}_{p_k}(\tilde{\mathcal{A}}^{2n_k}_y)^{-1} - \mathrm{Id}\| = \|(\tilde{\mathcal{A}}^{-2n_k}_{f^{2n_k}y})^{-1}\tilde{\mathcal{A}}^{-2n_k}_{f^{2n_k}p_k} - \mathrm{Id}\| \le K(K_2\delta_k)^\beta.$$

Since the cocycle $\tilde{\mathcal{A}}$ is isometric, it follows that

$$\|\tilde{\mathcal{A}}^{2n_k}_{f^{-n_k}(x)} - \tilde{\mathcal{A}}^{2n_k}_{p_k}\| \le \|\tilde{\mathcal{A}}^{2n_k}_{f^{-n_k}(x)} - \tilde{\mathcal{A}}^{2n_k}_y\| + \|\tilde{\mathcal{A}}^{2n_k}_y - \tilde{\mathcal{A}}^{2n_k}_{p_k}\| =$$
$$\|\tilde{\mathcal{A}}^{2n_k}_y\big((\tilde{\mathcal{A}}^{2n_k}_y)^{-1}\tilde{\mathcal{A}}^{2n_k}_{f^{-n_k}(x)} - \mathrm{Id}\big)\| + \|\big(\mathrm{Id} - \tilde{\mathcal{A}}^{2n_k}_{p_k}(\tilde{\mathcal{A}}^{2n_k}_y)^{-1}\big)\tilde{\mathcal{A}}^{2n_k}_y\| \le 2K(K_2\delta_k)^\beta.$$

Hence $\|\tilde{\mathcal{A}}^{2n_k}_{f^{-n_k}(x)} - \tilde{\mathcal{A}}^{2n_k}_{p_k}\| \to 0$ as $k \to \infty$.

We recall that in both cases (i) and (ii) the set $\{\tilde{C}(p)\}$ is bounded. Since $\tilde{C}(p_k)$ conjugates the periodic values of $\mathcal{A}$ and $\mathcal{B}$ at p_k, we have

$$\|\tilde{\mathcal{A}}^{2n_k}_{p_k} - \mathrm{Id}\| = \|\tilde{C}(p_k)^{-1}\tilde{B}^{2n_k}\tilde{C}(p_k) - \mathrm{Id}\| \le$$
$$\|\tilde{C}(p_k)^{-1}\| \cdot \|\tilde{C}(p_k)\| \cdot \|\tilde{B}^{2n_k} - \mathrm{Id}\| \le K_3\|\tilde{B}^{2n_k} - \mathrm{Id}\|.$$

Since $\tilde{B}^{2n_k} \to \mathrm{Id}$, it follows that $\tilde{\mathcal{A}}^{2n_k}_{p_k} \to \mathrm{Id}$. Thus using (3.5) we conclude that

$$(C^s(x))^{-1} \circ C^u(x) = \lim_k \tilde{\mathcal{A}}^{2n_k}_{f^{-n_k}(x)} = \mathrm{Id}, \quad \text{that is,} \quad C^s(x) = C^u(x).$$

For each $x \in S = W^s(q) \cap W^u(q)$ we define

$$\bar{C}(x) = C^s(x) = C^u(x).$$

The function $\bar{C}$ is a conjugacy between $\tilde{\mathcal{A}}$ and $\tilde{B}$ on the f-invariant homoclinic set S, which is known to be dense in X [Bo]. We will now show that the function $\bar{C}$ is Hölder continuous on S. Then it extends to a Hölder continuous function on X which conjugates the cocycles.

Let x and y be two sufficiently close points in S. We consider the point $z = W^u_{loc}(x) \cap W^s_{loc}(y)$, which is also in S. By the definition of $\bar{C} = C^s$

$$\bar{C}(z) = C^s(z) = H^s_{q,z} \qquad \bar{C}(y) = C^s(y) = H^s_{q,y},$$

and using properties of holonomies we obtain

$$\bar{C}(z) \circ \bar{C}(y)^{-1} = H^s_{q,z} \circ (H^s_{q,y})^{-1} = H^s_{q,z} \circ H^s_{y,q} = H^s_{y,z}. \tag{3.6}$$

Similarly, using unstable holonomies, we obtain $C(x) \circ C(z)^{-1} = H^u_{z,x}$ and hence

$$\bar{C}(x) \circ \bar{C}(y)^{-1} = C(x) \circ C(z)^{-1} \circ C(z) \circ C(y)^{-1} = H^u_{z,x} \circ H^s_{y,z}. \tag{3.7}$$

Since

$$\|H^u_{z,x} - \mathrm{Id}\| \le K \mathrm{dist}(x,z)^\beta, \quad \|H^u_{y,z} - \mathrm{Id}\| \le K \mathrm{dist}(y,z)^\beta, \quad \text{and}$$
$$\max\{\mathrm{dist}(x,z), \mathrm{dist}(y,z)\} \le K_1 \mathrm{dist}(x,y),$$

we conclude that

$$\|\bar{C}(x) \circ \bar{C}(y)^{-1} - \mathrm{Id}\| \le K' \mathrm{dist}(x,y)^\beta.$$

We note that since the cocycle $\tilde{\mathcal{A}}$ is isometric with respect to a Hölder continuous Riemannian metric, its holonomies $H^s_{x,y}$ and $H^u_{x,y}$ are uniformly bounded in x and y, and hence by definition $\|\bar{C}\|$ and $\|\bar{C}^{-1}\|$ are bounded on S by some constant M. Hence

$$\begin{aligned} d(\bar{C}(x), \bar{C}(y)) &= \|\bar{C}(x) - \bar{C}(y)\| + \|\bar{C}(x)^{-1} - \bar{C}(y)^{-1}\| \le \\ &\le \|\bar{C}(x)\bar{C}(y)^{-1} - \mathrm{Id}\| \cdot \|\bar{C}(y)\| + \|\bar{C}(x)^{-1}\| \cdot \|\mathrm{Id} - \bar{C}(x)\bar{C}(y)^{-1}\| \le \\ &\le 2MK' \,\mathrm{dist}(x,y)^\beta. \end{aligned} \tag{3.8}$$

We conclude that $\bar{C}$ is β-Hölder on S and hence extends to a β-Hölder continuous function on X which conjugates the cocycles. This completes the proof that $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ are β-Hölder cohomologous under the extra assumption that f has a fixed point.

Removing the fixed point assumption. Now we deduce the result in full generality, using the fact that f has periodic points of all sufficiently large periods.

Let p_1 be a periodic point of f of some period N. Then p_1 is the fixed point for f^N and we can apply the previous argument to the cocycles $\tilde{\mathcal{A}}^N$ and $\tilde{\mathcal{B}}^N$ over f^N. Hence there exists a β-Hölder conjugacy C_1 between $\mathcal{A}^N$ and $\mathcal{B}^N$. We want to show that there exists a conjugacy between the original cocycles $\mathcal{A}$ and $\mathcal{B}$ over f.

We define the *centralizer* of a $GL(d,\mathbb{R})$ cocycle $\mathcal{B}$ as the set of self-conjugacies

$$Z(\mathcal{B}) = \{D : X \to GL(d,\mathbb{R}) \mid \ \mathcal{B}_x = D(fx) \circ \mathcal{B}_x \circ D(x)^{-1} \quad \text{for all } x \in X\},$$

where all maps are considered in the β-Hölder category. It is easy to see that $Z(\mathcal{B})$ is a group with respect to pointwise multiplication. Let $\mathcal{C}(\mathcal{B},\mathcal{A})$ be the set of β-Hölder conjugacies between β-Hölder cocycles $\mathcal{A}$ and $\mathcal{B}$, and let $C_1 \in \mathcal{C}(\mathcal{B},\mathcal{A})$. Then $C_2 \in \mathcal{C}(\mathcal{B},\mathcal{A})$ if and only if $C_1 C_2^{-1} \in Z(\mathcal{B})$. Clearly $Z(\mathcal{B})$ is a subgroup of $Z(\mathcal{B}^k)$. It was shown in [S15, Proposition 4.8] that for any fiber bunched cocycle $\mathcal{B}$ there exists $M \ge 1$ such that

$$Z(\mathcal{B}^{MT}) = Z(\mathcal{B}^M) \ \text{ for all } \ T \ge 1.$$

Applying this to $\tilde{B}^N$ we see that there exists M such that

$$Z(\tilde{B}^{NM\cdot T}) = Z(\tilde{B}^{NM}) \text{ for all } T \geq 1.$$

We pick a periodic point p_2 of a period $K > 1$ relatively prime with MN. Using the argument with the fixed point for f^K we obtain a β-Hölder conjugacy C_2 between the cocycles $\tilde{\mathcal{A}}^K$ and $\tilde{B}^K$ over f^K. Since C_1 is a β-Hölder conjugacy between $\tilde{\mathcal{A}}^N$ and $\tilde{B}^N$, it is also a conjugacy between $\tilde{\mathcal{A}}^{NM}$ and $\tilde{B}^{NM}$. Similarly, both C_1 and C_2 are β-Hölder conjugacies between the cocycles $\tilde{\mathcal{A}}^{NMK}$ and $\tilde{B}^{NMK}$ over f^{NMK} and hence

$$C_1 C_2^{-1} \in Z(\tilde{B}^{NMK}) = Z(\tilde{B}^{NM}).$$

Now it follows that C_2 is also a conjugacy between the cocycles $\tilde{\mathcal{A}}^{NM}$ and $\tilde{B}^{NM}$. Thus C_2 is a conjugacy for the cocycles over f^{NM} and f^K, where MN and K are relatively prime. Hence there exist integers r and s such that $NMr + Ks = 1$, and it is easy to see that C_2 is also a conjugacy for the cocycles $\tilde{\mathcal{A}}$ and $\tilde{B}$ over f.

This completes the proof of the theorem. □

4. Proof of Theorem 1.9

First we establish that the conjugacy C between the derivative cocycle $\mathcal{A} = Df$ and the constant cocycle L is β-Hölder for some $0 < \beta < 1$. Then we use this to show that a conjugacy h between f and L is $C^{1+\beta}$. The bootstrap to C^∞ regularity follows immediately from [KSW25, Theorem 1.1].

4.1. Hölder continuity of the conjugacy between Df and L.

Let $0 < \rho_1 < \cdots < \rho_\ell$ be the distinct moduli of the eigenvalues of L, and let $\mathbb{R}^d = E^1 \oplus \cdots \oplus E^\ell$ be the corresponding Lyapunov splitting for L.

Suppose that the conjugacy C is continuous. Then

$$T\mathbb{T}^d = \mathcal{E} = \mathcal{E}^1 \oplus \cdots \oplus \mathcal{E}^\ell, \quad \text{where } \mathcal{E}^i = C(E^i), \tag{4.1}$$

is a continuous $\mathcal{A}$-invariant splitting. Moreover, for each i, the cocycle $\mathcal{A}_i = \mathcal{A}|\mathcal{E}^i$ has the same expansion/contraction rate as $L_i = L|E^i$. Thus the splitting is dominated for $\mathcal{A}$, and hence it is β-Hölder for some $0 < \beta < 1$. Also, since $\|\mathcal{A}_i^n\| \cdot \|(\mathcal{A}_i^n)^{-1}\|$ grows at most polynomially as for L_i, it follows that each $\mathcal{A}_i$ is fiber bunched. Then the conjugacy $C_i = C|E^i$ is β-Hölder by [KSW23, Theorem 2.1] as a measurable conjugacy between a fiber bunched cocycle and a constant cocycle with one exponent. So we conclude that C is β-Hölder.

Now suppose that the conjugacy C is measurable and f is C^1 close to L. Since f is a C^1-small perturbation of L, we have a continuous Df-invariant splitting $\oplus\mathcal{E}^i$ close to $\oplus E^i$ with similar expansion/contraction rates. In particular, it is dominated for $\mathcal{A}$ and hence it is β-Hölder for some $0 < \beta < 1$. Each $\mathcal{A}_i = \mathcal{A}|\mathcal{E}^i$ if fiber bunched as it is close to L_i. Also, we have $\mathcal{E}^i = C(E^i)$ almost everywhere since any measurable conjugacy preserves the Lyapunov exponents of vectors, see e.g. [KSW23, Lemma 4.4]. Hence each $C_i = C|E^i$ is a measurable conjugacy between $\mathcal{A}_i$ and L_i, and it is β-Hölder continuous as above by [KSW23, Theorem 2.1]. So we again conclude that C is β-Hölder.

4.2. **The main argument.** Let h be a topological conjugacy between L and f. Without loss of generality we can take h in the homotopy class of the identity. The stable and unstable foliations of f are topological foliations with $C^{1+\beta}$ leaves, moreover the leaves vary continuously in C^1 topology and their tangent bundles $\mathcal{E}^s$ and $\mathcal{E}^u$ are β-Hölder on $\mathbb{T}^d$. We say that such foliations have *uniformly $C^{1+\beta}$ leaves.*

We will now prove that h is a *uniformly $C^{1+\beta}$ diffeomorphism* along $\mathcal{W}^s$. By this we mean that its restrictions $h|\mathcal{W}^s(x)$ to the stable leaves are $C^{1+\beta}$ diffeomorphisms that depend continuously on x in C^1 topology and the derivative $D_x(h|\mathcal{W}^s(x))$ is β-Hölder on $\mathbb{T}^d$. Similarly, h is a uniformly $C^{1+\beta}$ diffeomorphism along $\mathcal{W}^u$, and then h is a $C^{1+\beta}$ diffeomorphism of $\mathbb{T}^d$ by Journé lemma [J88]. Let $1 \le m < \ell$ be such that

$$E^s = E^1 \oplus \cdots \oplus E^m$$

is the full stable sub-bundle for L. We denote by W^i the invariant linear foliations of L corresponding to Lyapunov sub-bundles E^i, and by $W^{j,m}$ the foliation corresponding to the sub-bundle $E^{j,m} = E^j \oplus \cdots \oplus \mathcal{E}^m$ where $1 \le j \le m$.

We use similar notations $\mathcal{E}^s = \mathcal{E}^1 \oplus \cdots \oplus \mathcal{E}^m$ for the Lyapunov splitting of f. By [DG24, Lemma 4.3] for each $1 \le j \le m$ the Lyapunov subbundle $\mathcal{E}^j$ and the weak stable subbundle

$$\mathcal{E}^{j,m} = \mathcal{E}^j \oplus \cdots \oplus \mathcal{E}^m$$

are tangent to f-invariant foliations with uniformly $C^{1+\beta}$ leaves, denoted by $\mathcal{W}^i$ and $\mathcal{W}^{j,m}$ respectively. Moreover, each weak foliation $\mathcal{W}^{j,m}$ is mapped by the conjugacy to the corresponding linear foliation for L, that is, $h(\mathcal{W}^{j,m}) = W^{j,m}$. In particular, we have $h(\mathcal{W}^m) = W^m$. The main part of the proof is the following proposition.

Proposition 4.1. *If h maps $\mathcal{W}^j$ to W^j for some j, then h is a uniformly $C^{1+\beta}$ diffeomorphism along $\mathcal{W}^j$.*

We use this proposition in an inductive process showing that h is a uniformly $C^{1+\beta}$ diffeomorphism along the weak foliations $\mathcal{W}^{j,m}$ for $j = m, \dots, 1$. The base case is given by applying the proposition with $j = m$, and in the end we obtain smoothness along the full stable $\mathcal{W}^{1,m} = \mathcal{W}^s$.

For the inductive step, we assume that h is a uniformly $C^{1+\beta}$ diffeomorphism along $\mathcal{W}^{j+1,m}$. Then we claim that the fast foliation $\mathcal{W}^{1,j}$, which always exists, is mapped to the corresponding linear one, $h(\mathcal{W}^{1,j}) = W^{1,j}$. This is given by implication (4) $\Longrightarrow$ (1) of [KS25, Theorem1.1], and specifically follows from [KS25, Proposition 5.1]. Its proof does not require regularity of f higher than uniformly $C^{1+\beta}$, and closeness of f to L is used only to obtain continuity of the splitting $\mathcal{E}^s = \mathcal{E}^1 \oplus \cdots \oplus \mathcal{E}^m$. Since L is weakly irreducible, the leaves of each Lyapunov foliation W^i are dense in $\mathbb{T}^d$, satisfying the assumption in [KS25]. This is the only place where weak irreducibility of L is used in the proof of Theorem 1.9.

Since we always have $h(\mathcal{W}^{j,m}) = W^{j,m}$, by intersecting we obtain that $h(\mathcal{W}^j) = W^j$. Now we apply Proposition 4.1 to concude that h is a uniformly $C^{1+\beta}$ diffeomorphism along $\mathcal{W}^j$. Together with the assumed smoothness along $\mathcal{W}^{j+1,m}$, this yields by Journé lemma that h is a uniformly $C^{1+\beta}$ diffeomorphism along $\mathcal{W}^{j,m}$ and completes the inductive step.

To complete the proof of Theorem 1.9 it remains to establish Proposition 4.1.

4.3. **Proof of Proposition 4.1.** We fix j and write

$$\mathcal{W} \text{ for } \mathcal{W}^j, \ W \text{ for } W^j, \ \mathcal{E} \text{ for } \mathcal{E}^j, \text{ and } E \text{ for } E^j.$$

We will use nonstationatry linearizations of f along $\mathcal{W}$ given by the following result.

Lemma 4.2. [K24, Corollary 4.8] (Non-stationary linearization).
Let f be a $C^{1+\beta}$, $0<\beta<1$, diffeomorphism of a compact manifold X, and let $\mathcal{W}$ be an f-invariant topological foliation of X with uniformly $C^{1+\beta}$ leaves. Suppose that for some continuous Riemannian metrics on $\mathcal{E}=T\mathcal{W}$ and constants $\gamma<1<\hat\gamma$ with $\hat\gamma\gamma^{1+\beta}<1$,

$$\hat\gamma^{-1} < \|Df_x(v)\| < \gamma \quad \textit{for any } x\in X \textit{ and any unit vector } v\in\mathcal{E}_x.$$

Then there exists a unique family $\{\varphi_x\}_{x\in X}$ of $C^{1+\beta}$ diffeomorphisms $\varphi_x : \mathcal{W}_x \to \mathcal{E}_x$ satisfying $\varphi_x(x)=0$ and $D_x\varphi_x = Id$ such that for each $x\in X$,

$$Df|\mathcal{E}_x = \varphi_{f(x)} \circ f \circ \varphi_x^{-1} : \mathcal{E}_x \to \mathcal{E}_{f(x)}. \tag{4.2}$$

The maps $\varphi_x|_{B^{\mathcal{W}}(x,R)}$ depend continuously on $x\in X$ in C^1 topology and have β-Hölder derivative with uniformly bounded Hölder constant.

Since the cocycle $\mathcal{A}=Df|\mathcal{E}$ is continuously conjugate to $L|E$, its quasiconformal distortion $\|\mathcal{A}_x^n\|\cdot\|(\mathcal{A}_x^n)^{-1}\|$ grows at most polynomially. Hence we can choose a continuous metric so that the distortion estimate $\gamma\hat\gamma$ is arbitrarily close to 1, and hence the bunching assumption $\hat\gamma\gamma^{1+\beta}<1$ is satisfied for any given $\beta>0$.

Since $h(\mathcal{W})=W$, the foliations $\mathcal{W}$ and $\mathcal{W}^u$ integrate to the joint foliation $\mathcal{V}=h^{-1}(V)$, where V is the linear foliation corresponding to the subspace $E\oplus E^u$. We will use holonomies of the foliation $\mathcal{W}^u$ inside $\mathcal{V}$ between the leaves of $\mathcal{W}$ and will denote them

$$\mathcal{H}_{x,y} : \mathcal{W}(x) \to \mathcal{W}(y) \quad \text{for any } y\in\mathcal{W}^u(x).$$

The maps $\mathcal{H}_{x,y}$ are globally defined, continuous, and depend continuously in C^0 topology on x and y when restricted to a ball of fixed size. We will now show that the maps $\mathcal{H}_{x,y}$ are differentiable and their derivatives coincide with the corresponding cocycle holonomies. For this we apply the following proposition to f^{-1}, so that γ and $\hat\gamma$ in it are the same as for f in Lemma 4.2. The second condition $\hat\gamma\gamma^{1+\beta}<1$ in (4.4) is the same as in Lemma 4.2 and ensures existence of the nonstationatry linearization. The first condition $\hat\gamma\gamma\,\nu^{\beta(1-\beta)}<1$ in (4.4) is stronger than $\hat\gamma\gamma\,\nu^{\beta}<1$, which is the s-bunching of $\mathcal{E}$ that ensures existence of s-holonomies of the β-Hölder cocycle $\mathcal{A}$. Both conditions are satisfied since $\gamma\hat\gamma$ can be chosen arbitrarily close to 1.

Lemma 4.3. (Smoothness of stable holonomies)
Let f be a $C^{1+\beta}$ diffeomorphism of a compact manifold X. Let $\mathcal{W}$ and $\mathcal{W}^s$ be transverse f-invariant topological foliations of X with uniformly $C^{1+\beta}$ leaves which integrate to a

joint topological foliation $\mathcal{V}$. Suppose that there are continuous Riemannian metrics on $\mathcal{E} = T\mathcal{W}$ and $\mathcal{E}^s = T\mathcal{W}^s$ and constants ν, γ, and $\hat{\gamma}$ such that

$$\begin{aligned} &\|Df_x(v^s)\| < \nu < 1 < \gamma^{-1} < \|Df_x(v)\| < \hat{\gamma} \\ &\text{for any } x \in X \text{ and any unit vectors } v^s \in \mathcal{E}^s_x \text{ and } v \in \mathcal{E}_x. \end{aligned} \tag{4.3}$$

Suppose that $\mathcal{E}$ is β-Hölder and f satisfies the bunching assumptions

$$\hat{\gamma}\gamma\, \nu^{\beta/(1+\beta)} < 1 \quad \text{and} \quad \hat{\gamma}\gamma^{1+\beta} < 1. \tag{4.4}$$

Then for any $x \in X$ and $y \in \mathcal{W}^s(x)$, the local $\mathcal{W}^s$ holonomy

$$\mathcal{H}_{x,y} : \mathcal{W}(x) \to \mathcal{W}(y) \text{ is differentiable and } D_x\mathcal{H}_{x,y} = H_{x,y},$$

where $H_{x,y} = H^{s,\mathcal{A}}_{x,y}$ is the $\mathcal{W}^s$ holonomy of the cocycle $\mathcal{A} = Df|\mathcal{E}$.

If f and the leaves of $\mathcal{V}$ were $C^{2+\beta}$, this lemma could be obtained using the C^r Section Theorem. This result seems new even for $C^{1+\beta}$ Anosov diffeomorphisms, yielding $C^{1+\beta}$ regularity of the stable holonomies and the stable foliation if f is close to conformal on $\mathcal{E}^u$, or if $\mathcal{E}^u$ is one dimensional.

Proof of Lemma 4.3. The first bunching assumption in (4.4) yields that

$$\hat{\gamma}\gamma\, \nu^{\beta} < \hat{\gamma}\gamma\, \nu^{\beta/(1+\beta)} < \theta \quad \text{for some } \theta < 1. \tag{4.5}$$

In particular, the cocycle $\mathcal{A} = Df|\mathcal{E}$ is s-bunched and hence it has β-Hölder s-holonomies $H_{x,y} = H^{s,\mathcal{A}}_{x,y} : \mathcal{E}_x \to \mathcal{E}_y$. The second bunching assumption in (4.4) yields existence of nonstationary linearizations $\{\varphi_x\}_{x\in X}$ of f along $\mathcal{W}$ by Lemma 4.2.

We fix a small $0 < \varepsilon_0 < 1$ so that for any $x \in X$ and $y \in \mathcal{W}^s(x)$ with $d_{\mathcal{W}^s}(x, y) < \varepsilon_0/2$ the holonomy $\mathcal{H}_{x,y}$ of $\mathcal{W}^s$ is defined and satisfies $d_{\mathcal{W}^s}(z, \mathcal{H}_{x,y}(z)) < \varepsilon_0$ on the ball of radius ε_0 in $\mathcal{W}(x)$ centered at x. We fix such x and y and lift the holonomy $\mathcal{H}_{x,y}$ to $\mathcal{E}_x = T_x\mathcal{W}$ using the nonstationatry linearizations:

$$\bar{\mathcal{H}}_{x,y} = \varphi_{f(x)} \circ \mathcal{H}_{x,y} \circ \varphi_x^{-1} : \mathcal{E}_x \to \mathcal{E}_{f(x)}.$$

Since $D_x\varphi_x = \mathrm{Id} = D_y\varphi_y$ it suffices to show that $D_x\bar{\mathcal{H}}_{x,y} = H_{x,y}$. We will prove that

$$\Delta(t) = \bar{\mathcal{H}}_{x,y}(t) - H_{x,y}(t) = o(t), \quad \text{where } t \in \mathcal{E}_x.$$

We iterate forward and denote $x_n = f^n(x)$, $y_n = f^n(y)$, and

$$\Delta_n = \bar{\mathcal{H}}_{x_n,y_n} - H_{x_n,y_n}.$$

Using invariance properties of holonomies (3.1) and linearizations (4.2), and denoting $t_n = \mathcal{A}^n_x(t)$, we can write

$$\Delta(t) = (\mathcal{A}^n_y)^{-1} \circ \Delta_n \circ \mathcal{A}^n_x(t) = (\mathcal{A}^n_y)^{-1}\Delta_n(t_n).$$

Using (4.3) we obtain

$$\|t_n\| \le \hat{\gamma}^n\|t\| \quad \text{and} \quad \|\Delta(t)\| \le \gamma^n\|\Delta_n(t_n)\|.$$

Now we estimate Δ.

$$\|\Delta(t)\| \le \gamma^n\|\Delta_n(t_n)\| \le \gamma^n\|H_{x_n,y_n}(t_n) - t_n\| + \gamma^n\|\bar{\mathcal{H}}_{x_n,y_n}(t_n) - t_n\|. \tag{4.6}$$

For the first term we use the Hölder property of cocycle holonomies (3.2).

$$\|H_{x_n,y_n}(t_n) - t_n\| \;\le\; \|H_{x_n,y_n} - \mathrm{Id}\|\,\|t_n\| \;\le\; K d(x_n,y_n)^\beta\,\hat\gamma^n\,\|t\| \;\le\; K\varepsilon_0^\beta \nu^{n\beta}\,\hat\gamma^n\,\|t\|,$$

and so using (4.5) we obtain

$$\gamma^n\|H_{x_n,y_n}(t_n) - t_n\| \;\le\; K\gamma^n\hat\gamma^n\,\nu^{n\beta}\,\|t\| \;<\; K\theta^n\|t\|. \tag{4.7}$$

Now we estimate the second term in (4.6). Denoting

$$z_n = \varphi_{x_n}^{-1}(t_n) \in \mathcal{W}(x_n) \quad \text{and} \quad w_n = \mathcal{H}_{x_n,y_n}(z_n) \in \mathcal{W}(y_n)$$

we have

$$\bar{\mathcal{H}}_{x_n,y_n}(t_n) = \varphi_{y_n} \circ \mathcal{H}_{x_n,y_n} \circ \varphi_{x_n}^{-1}(t_n) = \varphi_{y_n}(w_n).$$

Using local coordinates we identify a small ball around x_n with a ball in $\mathbb{R}^d$, where x_n is identified with 0. Then

$$\|\bar{\mathcal{H}}_{x_n,y_n}(t_n) - t_n\| \le \|t_n - z_n\| + \|z_n - w_n\| + \|w_n - \varphi_{y_n}(w_n)\|. \tag{4.8}$$

For the first term we have

$$\|t_n - z_n\| = \|t_n - \varphi_{x_n}^{-1}(t_n)\| \le M\|t_n\|^{1+\beta} \le M(\hat\gamma^n\|t\|)^{1+\beta}$$

since $\varphi_x(x) = 0$, $D_x\varphi_x = \mathrm{Id} = D_0\varphi_x^{-1}$, and $C^{1+\beta}$ norms of φ_{x_n} and $\varphi_{x_n}^{-1}$ on small balls are uniformly bounded. The middle term in (4.8) is estimated by contraction of $\mathcal{W}^s$ as

$$\|z_n - w_n\| \le \nu^n\varepsilon_0.$$

The middle term bound decays while the first term bound grows with n, and we choose $n = n(t) \in \mathbb{N}$ to be the first for which

$$\nu^n\varepsilon_0 < M(\hat\gamma^n\|t\|)^{1+\beta}.$$

For any sufficiently small t, this choice ensures that $n = n(t)$ is large and hence both that $\nu^n\varepsilon_0$ and $M(\hat\gamma^n\|t\|)^{1+\beta}$ are small. This also yields that $\hat\gamma^n\|t\|$ is small, and so we can assume that $M(\hat\gamma^n\|t\|)^{1+\beta} \le \hat\gamma^n\|t\|$. Then we have

$$\begin{aligned}\|y_n - w_n\| &\le \|y_n - 0\| + \|0 - t_n\| + \|t_n - z_n\| + \|z_n - w_n\| \\ &\le \nu^n\varepsilon_0 + \hat\gamma^n\|t\| + M(\hat\gamma^n\|t\|)^{1+\beta} + \nu^n\varepsilon_0 \;\le\; 4\hat\gamma^n\|t\|.\end{aligned}$$

Now the third term in (4.8) can be estimated similarly to first one by

$$\|w_n - \varphi_{y_n}(w_n)\| \le M(\|w_n - y_n\|)^{1+\beta} \le M(4\hat\gamma^n\|t\|)^{1+\beta} \le 16M(\hat\gamma^n\|t\|)^{1+\beta}.$$

We conclude that

$$\|\bar{\mathcal{H}}_{x_n,y_n}(t_n) - t_n\| \le 18M(\hat\gamma^n\|t\|)^{1+\beta}.$$

Using this and (4.7) in (4.6) we obtain

$$\|\Delta(t)\| \le K\theta^n\|t\| + \gamma^n 18M(\hat\gamma^n\|t\|)^{1+\beta},$$

and hence

$$\|\Delta(t)\|\,\|t\|^{-1} \le K\theta^n + 18M(\gamma\hat\gamma^{1+\beta})^n\|t\|^\beta. \tag{4.9}$$

From the choice of $n = n(t)$ we have the following exponential estimate for $\|t\|$

$$\nu^n \sim (\hat{\gamma}^n \|t\|)^{1+\beta} \iff \|t\| \sim (\nu^{1/(1+\beta)}/\hat{\gamma})^n.$$

Hence the last term in (4.9) can be bounded using (4.5) as

$$18M(\gamma\hat{\gamma}^{1+\beta})^n \|t\|^\beta \le M'(\gamma\hat{\gamma}^{1+\beta})^n \cdot (\nu^{1/(1+\beta)}/\hat{\gamma})^{\beta n} = M'(\gamma\hat{\gamma}\,\nu^{\beta/(1+\beta)})^n < M'\theta^n.$$

We conclude that for $n = n(t)$,

$$\|\Delta(t)\|\,\|t\|^{-1} < (K + M')\theta^n.$$

Hence for any $\varepsilon > 0$ there is $\delta > 0$ such that $\|t\| < \delta$ implies that $n = n(t)$ is large enough and $(K + M')\theta^n < \varepsilon$. This proves that $\|\Delta(t)\| = o(t)$ and completes the proof of the Lemma 4.3. $\square$

We recall that C is a β-Hölder conjugacy between β fiber bunched cocycles $\mathcal{A} = Df|\mathcal{E}$ and $L|E$. By [S15, Proposition 4.5], C intertwines the u-holonomy $H^{u,\mathcal{A}}_{x,y} : \mathcal{E}_x \to \mathcal{E}_y$ of $\mathcal{A}$ with the trivial holonomy $\mathrm{Id} : E_x \to E_y$ of the constant cocycle $L|E$, that is,

$$H^{u,\mathcal{A}}_{x,y} = C(y) \circ \mathrm{Id} \circ C(x)^{-1}, \quad \text{and so} \quad H^{u,\mathcal{A}}_{x,y} \circ C(x) = C(y). \tag{4.10}$$

We obtain a β-Hölder Riemannian metric g on the bundle $\mathcal{E} = T\mathcal{W}$ by pushing a constant Euclidean metric on E by C. It is clear from the formula above that the holonomy $H^{u,\mathcal{A}}_{x,y}$ is an isometry with respect to g.

We equip the leaves of $\mathcal{W}$ with the Riemannian metric g. Now Proposition 4.3, applied to $\mathcal{W}^u$ and $\mathcal{W}$ with f^{-1}, yields that the derivative $D_x\mathcal{H}_{x,y}$ of the holonomy $\mathcal{H}_{x,y} : \mathcal{W}(x) \to \mathcal{W}(y)$ of $\mathcal{W}^u$ coincides with $H^{u,\mathcal{A}}_{x,y}$, and thus it is an isometry with respect to g. We conclude that $\mathcal{H}_{x,y} : (\mathcal{W}(x), g) \to (\mathcal{W}(y), g)$ is an isometry.

We fix an arbitrary $x \in \mathbb{T}^d$ and take any $z \in \mathcal{W}(x)$. Since the linear unstable leaf $W^u(x)$ is dense in $\mathbb{T}^d$, there exists a sequence of vectors $v_n \in E^u \subset \mathbb{R}^d$ such that $h(x) + v_n \in \mathbb{T}^d$ converges to $h(z)$. Denoting $y_n = h^{-1}(h(x) + v_n)$ we obtain a sequence of points $y_n \in \mathcal{W}^u(x)$ converging to z. The linear holonomies $\hat{\mathcal{H}}_{h(x),\,h(y_n)}$ of W^u are given by translations by v_n, and hence they converge in C^0 to the translation T_v in $W(h(x))$ by the vector $v = h(z) - h(x)$. Hence the holonomies $\mathcal{H}_{x,y_n}$ also converge in C^0 norm to the corresponding homeomorphism

$$\mathcal{T}_v : \mathcal{W}(x) \to \mathcal{W}(x) \;\text{ given by }\; \mathcal{T}_v = h^{-1} \circ T_v \circ h.$$

Since the holonomies $\mathcal{H}_{x,y_n}$ are isometries between $\mathcal{W}(x)$ and $\mathcal{W}(y_n)$, the limit $\mathcal{T}_v$ is an isometry of $(\mathcal{W}(x), g)$. Thus $h_x = h|\mathcal{W}(x)$ conjugates the action of $E = \mathbb{R}^k$ by translations of $W(h(x))$ with the corresponding continuous action of $\mathbb{R}^k$ by isometries $\mathcal{T}_v$ of $\mathcal{W}(x)$. Denoting the group of isometries of $\mathcal{W}(x)$ by G_x we obtain an injective continuous homomorphism

$$\eta_x : E \to G_x \quad \text{given by } \; \eta_x(v) = \mathcal{T}_v = (h_x)^{-1} \circ T_v \circ h_x.$$

Since the Riemannian metric g is β-Hölder, the elements of G_x are $C^{1+\beta}$ diffeomorphisms [T06]. Classical results imply that G_x is a finite dimensional Lie group.

In our case this can be seen directly as follows. We claim that the $C^{1+\beta}$ nonstationary linearization $\varphi_x : (\mathcal{W}(x), g) \to (\mathcal{E}_x, g_x)$ is an isometry, giving a natural $C^{1+\beta}$ identification of G_x with the Euclidean group of $\mathcal{E}_x$. Indeed, it is easy to check that the family of maps $\tilde{H}_{z,x} = D_z\varphi_x : \mathcal{E}_z \to \mathcal{E}_x = T_{\varphi(z)}\mathcal{E}_x$ is a β-Hölder holonomy for cocycle $\mathcal{A} = Df|\mathcal{E}$ along the foliation $\mathcal{W}$. As in (4.10) above, by [S15, Proposition 4.5] the conjugacy C intertwines this (unique) holonomy for $\mathcal{A}$ with the trivial holonomy for $L|E$, and hence $D_z\varphi_x = \tilde{H}_{z,x} : (\mathcal{E}_z, g_z) \to (\mathcal{E}_x, g_x)$ is an isometry for each $z \in \mathcal{W}(x)$.

Any continuous homomorphism between Lie groups is a C^∞ Lie group homomorphism, see for example [Ha, Corollary 3.50]. Hence η_x is a C^∞ diffeomorphism onto its image in G_x. Since $(h_x)^{-1}$ is determined by η_x as

$$(h_x)^{-1}(h(x) + v) = \mathcal{T}_v(0) = \eta_x(v)(0),$$

we obtain that $(h_x)^{-1}$ is a $C^{1+\beta}$ diffeomorphism between $W(h(x))$ and $\mathcal{W}(x)$. Hence $h_x = h|\mathcal{W}(x)$ is also a $C^{1+\beta}$ diffeomorphism. Since the homomorphisms η_x depend continuously on x, so do their derivatives, which yields that h is uniformly C^1 along $\mathcal{W}$. Now the derivative $\tilde{C}(x) = D_x(h_x)$ is a continuous conjugacy between cocycles $\mathcal{A}$ and $L|E$ and hence is β-Hölder on $\mathbb{T}^d$ by [KSW23, Theorem 2.1]. This shows that h is uniformly $C^{1+\beta}$ along $\mathcal{W}$ and completes the proof of Proposition 4.1. □

References

[Bac15] L. Backes. *Rigidity of fiber bunched cocycles.* Bul. Brazilian Math. Soc., Vol. 46, Issue 2 (2015), 163-179.

[Bo] R. Bowen. *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms.* Lecture Notes in Mathematics, Vol. 470, Springer-Verlag, 1975.

[D25] J. DeWitt. *Periodic data rigidity of Anosov automorphisms with Jordan blocks.* Ergodic Theory and Dynamical Systems. Vol. 45, Issue 4 (2025), 1129-1160.

[DG24] J. DeWitt, A. Gogolev. *Dominated splitting from constant periodic data and global rigidity of Anosov automorphisms.* Geometric and Functional Analysis 34 (2024): 1370-1398.

[F69] J. Franks. *Anosov diffeomorphisms on tori.* Transactions of the AMS, *145* (1969), 117-124.

[G08] A. Gogolev. *Smooth conjugacy of Anosov diffeomorphisms on higher dimensional tori.* Journal of Modern Dynamics, 2, no. 4, 645-700. (2008)

[GKS11] A. Gogolev, B. Kalinin, V. Sadovskaya. *Local rigidity for Anosov automorphisms.* Math. Research Letters, 18 (2011), no. 05, 843-858.

[Ha] B. Hall. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction.* Graduate Texts in Mathematics, 222, 2nd ed., (2015) Springer.

[J88] J.-L. Journé. *A regularity lemma for functions of several variables.* Revista Matemática Iberoamericana 4 (1988), no. 2, 187-193.

[K11] B. Kalinin. *Livšic theorem for matrix cocycles.* Annals of Math., 173 (2011), no. 2, 1025-1042.

[K24] B. Kalinin. *Non-stationary normal forms for contracting extensions.* A Vision for Dynamics in the 21st Century, pp. 207-231, Cambridge University Press (2024).

[KS10] B. Kalinin, V. Sadovskaya. *Linear cocycles over hyperbolic systems and criteria of conformality.* Journal of Modern Dynamics, vol. 4 (2010), no. 3, 419-441.

[KS13] B. Kalinin, V. Sadovskaya. *Cocycles with one exponent over partially hyperbolic systems.* Geometriae Dedicata, Vol. 167, Issue 1 (2013), 167-188.

[KS14] B. Kalinin, V. Sadovskaya. *Holonomies and cohomology for cocyces over partially hyperbolic diffeomorphisms.* Discrete and Continuous Dynamical Systems. Vol. 36, Issue 1 (2016), 245-259.

[KS25] B. Kalinin, V. Sadovskaya. *Rigidity of strong and weak foliations.* Preprint.

[KSW23] B. Kalinin, V. Sadovskaya, Z. Wang. *Smooth local rigidity for hyperbolic toral automorphisms.* Communication of the AMS, Vol. 3 (2023), 290-328.

[KSW25] B. Kalinin, V. Sadovskaya, Z. Wang. *Global smooth rigidity for toral automorphisms.* Submitted.

[KtH] A. Katok and B. Hasselblatt. *Introduction to the modern theory of dynamical systems.* Encyclopedia of Mathematics and its Applications 54, Cambridge University Press, 1995.

[Li71] A. N. Livšic. *Homology properties of Y-systems.* Math. Zametki 10, 758-763, 1971.

[Li72] A. N. Livšic. *Cohomology of dynamical systems.* Math. USSR Izvestija 6, 1278-1301, 1972.

[L87] R. de la Llave. *Invariants for smooth conjugacy of hyperbolic dynamical systems II.* Commun. Math. Phys., 109 (1987), 368-378.

[L92] R. de la Llave. *Smooth conjugacy and SRB measures for uniformly and nonuniformly hyperbolic systems.* Communications in Mathematical Physics, **150** (1992), 289-320.

[L02] R. de la Llave. *Rigidity of higher dimensional conformal Anosov systems.* Ergodic Theory and Dynamical Systems **22** (2002), no. 6, 1845-1870.

[LM88] R. de la Llave, R. Moriyón. *Invariants for smooth conjugacy of hyperbolic dynamical systems IV.* Commun. Math. Phys., 116 (1988), 185-192.

[M73] A. Manning. *Anosov diffeomorphisms on nilmanifolds.* Proc. Amer. Math. Soc. 38 (1973), 423–426.

[Pa99] W. Parry. *The Livšic periodic point theorem for non-Abelian cocycles.* Ergodic Theory and Dynamical Systems, 19(3), 687-701, 1999.

[PaP97] W. Parry, M. Pollicott. *The Livšic cocycle equation for compact Lie group extensions of hyperbolic systems.* J. London Math. Soc. (2), 56(2) 405-416, 1997.

[S13] V. Sadovskaya. *Cohomology of $GL(2,\mathbb{R})$-valued cocycles over hyperbolic systems.* Discrete and Continuous Dynamical Systems, vol. 33, no. 5 (2013), 2085-2104.

[S15] V. Sadovskaya. *Cohomology of fiber bunched cocycles over hyperbolic systems.* Ergodic Theory and Dynamical Systems, Vol. 35, Issue 8 (2015), 2669-2688.

[S17] V. Sadovskaya. *Fiber bunching and cohomology for Banach cocycles over hyperbolic systems.* Discrete and Continuous Dynamical Systems, vol. 37, no. 9 (2017), 4959-4972.

[Schm99] K. Schmidt. *Remarks on Livšic theory for non-Abelian cocycles.* Ergodic Theory and Dynamical Systems, 19(3), (1999), 703-721.

[T06] M. Taylor. *Existence and Regularity of Isometries.* Trans. Amer. Math. Soc., Vol. 358, No. 6 (2006), 2415-2423.

[Wa70] P. Walters. *Conjugacy properties of affine transformations of nilmanifolds.* Math. Systems Theory 4 (1970), 327-333.

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