LINEAR COCYCLES OVER HYPERBOLIC SYSTEMS

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Dedicated to the memory of Anatole Katok

1. INTRODUCTION

Cocycles are a fundamental tool in the study of dynamical systems and group actions. In smooth dynamics, the differential and related objects provide important examples of cocycles. For a hyperbolic or partially hyperbolic diffeomorphism f of a manifold X, the restrictions of the differential to invariant sub-bundles of TX, such as stable, unstable, and center, give examples of so called linear cocycles. Linear cocycles on trivial bundles can be viewed as $GL(d, \mathbb{R})$-valued cocycles. Another class of examples is given by matrix or operator-valued cocycles over non-smooth hyperbolic systems such as shifts and subshifts of finite type. Locally constant cocycles over these systems correspond to random and Markovian sequences of matrices and operators.

Cohomology, or conjugacy, is a notion of equivalence for cocycles, which can be considered in various regularity classes. The key questions in the study of cocycles include:

- When two cocycles are cohomologous;
- When a measurable conjugacy between two cocycles is continuous;
- When a cocycle is cohomologous to a simpler one, for example to the identity cocycle or to one with values in a smaller group such as orthogonal or conformal.

Reduction to a smaller group can often be interpreted as existence and regularity of an invariant geometric structure, such as a Riemannian metric or a conformal structure. As abundance of periodic orbits is one of the key features of hyperbolic systems, one may try to deduce properties of cocycles from their periodic data, that is, their products along the periodic orbits of f. Questions include whether conformality at the periodic points implies conformality of the cocycle, and whether conjugacy of the periodic data for two cocycles implies cohomology.

Another important problem in the study of cocycles is estimating their growth and, in particular, the asymptotic growth rates given by the Lyapunov exponents. For hyperbolic systems, approximations of the Lyapunov exponents and growth estimates can sometimes be obtained from the periodic data of the cocycle. Cocycles with one Lyapunov exponent give a generalization of conformal ones.

Many of the results and techniques for hyperbolic systems extend to the partially hyperbolic setting, and periodic approximation of Lyapunov exponents also holds in the non-uniformly hyperbolic case.

The motivation for studying linear cocycles comes in part from the area of local and global rigidity of hyperbolic systems. Global rigidity refers to existence of a smooth conjugacy to an algebraic model, and local rigidity is smoothness of the conjugacy to a C^1-small perturbation when natural obstructions vanish. The results on linear cocycles produced many applications to this area.

The study of linear cocycles over hyperbolic and partially hyperbolic systems is an active area, and a variety of questions have been explored. In this survey we focus on the ones mentioned above. Other directions include the study of hyperbolicity, simplicity of Lyapunov spectrum, and (dis)continuous dependence of Lyapunov exponents on the cocycle. Many results in these areas are related to the invariance principle for cocycles developed in [V08, AV10, ASV13].

The text is organized as follows. In Section 2 we give definitions of the main classes of hyperbolic dynamical systems, and of linear and Banach cocycles over these systems. In Section 3 we discuss fiber bunching and holonomies of a cocycle, which are used in subsequent sections. Section 4 is devoted to the three questions on cohomology of group-valued cocycles over hyperbolic systems mentioned above. We state several results, give examples, and outline some of the proofs and constructions. In Section 5 we discuss the Lyapunov exponents of a cocycle and their periodic approximation. We also deduce growth estimates for cocycles. In Section 6 we consider conformal structures and their relation to uniform quasiconformality of a linear cocycle. We also obtain conformality and isometry for a linear cocycle from its periodic data, and we state an analogous result for the infinite-dimensional setting. In Section 7 we discuss cocycles with one Lyapunov exponent, and we show that they have a specific structure. We also give a classification of $GL(2, \mathbb{R})$-valued cocycles with one exponent. Section 8 is devoted to cocycles over partially hyperbolic diffeomorphisms and extensions of some of the results for hyperbolic systems to this setting. In Section 9 we give some applications of results on linear cocycles to the questions of rigidity for hyperbolic diffeomorphisms and flows.

2. Cocycles over hyperbolic dynamical systems

We will consider cocycles over *hyperbolic* dynamical systems. Below we describe the three main classes of such systems, see [KtH] for more details.

Transitive Anosov diffeomorphisms. A diffeomorphism f of a compact connected manifold X is called *Anosov* if there exist a splitting of the tangent bundle TX into a direct sum of two Df-invariant continuous sub-bundles E^s and E^u, a Riemannian metric on X, and continuous functions ν and $\hat{\nu}$ such that

$$(2.1) \qquad \|Df_x(v^s)\| < \nu(x) < 1 < \hat{\nu}(x) < \|Df_x(v^u)\|$$

for any $x \in X$ and unit vectors $v^s \in E^s(x)$ and $v^u \in E^u(x)$. The sub-bundles E^s and E^u are called stable and unstable. We assume that f is at least $C^{1+\text{Hölder}}$, and then the sub-bundles are Hölder continuous.

The stable and unstable sub-bundles are tangent to the stable and unstable foliations W^s and W^u respectively. We define the *local stable manifold* of x, $W^s_{\text{loc}}(x)$, as a ball

centered at x of radius ρ in the intrinsic metric of $W^s(x)$. We choose ρ sufficiently small so that $W^s_{\mathrm{loc}}(x) \cap W^u_{\mathrm{loc}}(z)$ consists of a single point for any sufficiently close x and z in X, and for every $x \in X$ we have $\|Df|_{E^s(y)}\| \le \nu(x)$ for all y in $W^s_{\mathrm{loc}}(x)$. The first property is called the local product structure of the foliations, and the second one implies that

$$(2.2) \qquad \mathrm{dist}(f(x), f(y)) \le \nu(x) \cdot \mathrm{dist}(x, y) \quad \text{for all } x \in X \text{ and } y \in W^s_{\mathrm{loc}}(x).$$

Local unstable manifolds are defined similarly.

A diffeomorphism f is *(topologically) transitive* if there is a point x in X with dense orbit. All known examples of Anosov diffeomorphisms have this property. We note that any transitive Anosov diffeomorphism is *topological mixing,* that is, for any open sets U_1 and U_2 in X, there exists $N \in \mathbb{N}$ such that $f^n(U_1) \cap U_2 \ne \emptyset$ for all $n \ge N$.

Topologically mixing diffeomorphisms of locally maximal hyperbolic sets. More generally, let f be a diffeomorphism of a manifold $\mathcal{M}$. A compact f-invariant set $X \subseteq \mathcal{M}$ is called *hyperbolic* if there exist a continuous Df-invariant splitting $T_X \mathcal{M} = E^s \oplus E^u$, and a Riemannian metric and continuous functions ν, $\hat{\nu}$ on an open set $U \supseteq X$ such that (2.1) holds for all $x \in X$. The set X is called *locally maximal* if $X = \bigcap_{n \in \mathbb{Z}} f^{-n}(U)$ for some open set U containing X.

Mixing subshifts of finite type. Let M be $k \times k$ matrix with entries from $\{0, 1\}$ such that all entries of M^N are positive for some N. Let

$$X = \{\, x = (x_n)_{n \in \mathbb{Z}} \; : \; 1 \le x_n \le k \;\; \text{and} \;\; M_{x_n, x_{n+1}} = 1 \text{ for every } n \in \mathbb{Z} \,\}.$$

The shift map $f : X \to X$ is defined by $(fx)_n = x_{n+1}$. The system (X, f) is called a *mixing subshift of finite type.* We fix $\nu \in (0, 1)$ and consider the metric

$$\mathrm{dist}(x, y) = d_\nu(x, y) = \nu^{n(x,y)}, \quad \text{where} \;\; n(x, y) = \min\{\, |i| \; : \; x_i \ne y_i \}.$$

The following sets play the role of the local stable and unstable manifolds of x:

$$W^s_{\mathrm{loc}}(x) = \{\, y \mid x_i = y_i, \;\; i \ge 0 \,\}, \quad W^u_{\mathrm{loc}}(x) = \{\, y \mid x_i = y_i, \;\; i \le 0 \,\},$$

and we can take $\nu(x) = \nu$ and $\hat{\nu}(x) = \nu^{-1}$.

Lemma 2.1 (Anosov Closing Lemma)**.** [KtH, 6.4.15-17] *Let (X, f) be a hyperbolic system. Then there exist constants $c, \delta_0 > 0$ such that for any $x \in X$ and $k \in \mathbb{N}$ with $\mathrm{dist}(x, f^k x) < \delta_0$ there exists a periodic point $p \in X$ with $f^k p = p$ such that the orbit segments $x, fx, \dots, f^k x$ and $p, fp, \dots, f^k p$ remain close:*

$$\mathrm{dist}(f^i x, f^i p) \le c \, \mathrm{dist}(x, f^k x) \quad \text{for every } i = 0, \dots, k.$$

Moreover,

$$(2.3) \qquad \mathrm{dist}(f^i x, f^i p) \le c \, \mathrm{dist}(x, f^k x)\, e^{-\gamma \min\{\, i,\, k-i \}} \quad \text{for every } i = 0, \dots, k,$$

where $e^{-\gamma} = \max\limits_{x \in X} \, \max\{\nu(x), (\hat{\nu}(x))^{-1}\} < 1$.

Linear and Banach cocycles over dynamical systems. Let G be a topological group with a complete metric. We primarily consider the group $G = GL(d, \mathbb{R})$ and

more generally $GL(V)$, the group of invertible bounded linear operators on a Banach space V. We fix the metric d on the group:

$$d(A, B) = \|A - B\| + \|A^{-1} - B^{-1}\|, \quad \text{where } \| \, . \, \| \text{ is the operator norm.}$$

Definition 2.2. *Let A be a function from X to a group G. The G-valued cocycle over (X, f) generated by A is the map $\mathcal{A} : X \times \mathbb{Z} \to G$ defined by $\mathcal{A}(x, 0) = Id$ and*

$$\mathcal{A}(x, n) = \mathcal{A}_x^n = A(f^{n-1}x) \circ \cdots \circ A(x) \;\; and \;\; \mathcal{A}(x, -n) = \mathcal{A}_x^{-n} = (\mathcal{A}_{f^{-n}x}^n)^{-1} \; for \; n \in \mathbb{N}.$$

Clearly, $\mathcal{A}$ satisfies the cocycle equation $\mathcal{A}_x^{n+k} = \mathcal{A}_{f^k x}^n \circ \mathcal{A}_x^k$.

We refer to cocycles with values in $G = GL(d, \mathbb{R})$ and in $GL(V)$ as *linear cocycles* and *Banach cocycles*, respectively.

A prime example of a linear cocycle is the *derivative cocycle*, i.e., the differential of a diffeomorphism $f : X \to X$, where the tangent bundle of X is trivial, i.e. $TX = X \times \mathbb{R}^d$. Then Df can be viewed as a $GL(d, \mathbb{R})$-valued cocycle:

$$A(x) = \mathcal{A}_x = D_x f \in GL(d, \mathbb{R}) \quad \text{and} \quad \mathcal{A}_x^n = D_x f^n.$$

One can also consider restrictions of Df to invariant sub-bundles of TX, for example the stable and unstable sub-bundles.

Another important class of examples is given by random and Markovian sequences of matrices and operators. They correspond to locally constant cocycles over full shifts or subshifts of finite type.

Standing assumption. *We will always assume that the cocycle $\mathcal{A}$ is β-Hölder, that is, its generator A is a Hölder continuous function with exponent β from X to G.*

Hölder continuity of the cocycle is natural in view of the examples above. The derivative cocycle of a $C^{1+\text{Hölder}}$ diffeomorphism f is Hölder continuous, and so are its restrictions to the stable and unstable sub-bundles. These subbundles are usually only Hölder continuous even for a more regular diffeomorphism [HaW99], and hence $Df|_{E^s}$ and $Df|_{E^s}$ are also only Hölder continuous. Also, symbolic dynamical systems have a Hölder structure, but no smooth one. In addition, Hölder continuity of the cocycle is necessary to develop a meaningful theory, even for $\mathbb{R}$-valued cocycles over hyperbolic systems.

Bundle setting. The derivative cocycle example suggests viewing a linear cocycle $\mathcal{A}$ as an automorphism of a vector bundle $\mathcal{E} = X \times \mathbb{R}^d$ with the fiber map $\mathcal{A}_x : \mathcal{E}_x \to \mathcal{E}_{fx}$.

More generally, one can allow non-trivial bundles, and most of our results extend to this setting. Let (X, f) be a hyperbolic system and $P : \mathcal{E} \to X$ be a finite dimensional β-Hölder vector bundle over X. A continuous linear cocycle over f is a homeomorphism $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ such that

$$P \circ \mathcal{A} = f \circ P \quad \text{and } \mathcal{A}_x : \mathcal{E}_x \to \mathcal{E}_{fx} \text{ is a linear isomorphism.}$$

Such an $\mathcal{A}$ is called β-Hölder if $\mathcal{A}_x$ depends β-Hölder on x, with proper identification of fibers at nearby points. We refer to Section 2.2 of [KS13] for a detailed description of this setting.

3. Fiber bunching and holonomies

In the study of cocycles, comparing their iterates along exponentially converging orbits plays an important role. If the group G is Abelian, the products of the form $(\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n$, where $y \in W^s(x)$, can be rearranged as $(\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n = \prod_{i=0}^{n-1} (A(f^i(y))^{-1} \circ A(f^i(x)))$ and then estimated easily using exponential closeness of the iterates of x and y and Hölder property of the cocycle. In the non-commutative case, estimating such products is more difficult and various assumptions on the growth of the cocycle have been used to ensure their convergence. For linear and Banach cocycles, we use a condition called *fiber bunching*, which means that non-conformality of the cocycle in the fiber is dominated by the expansion and contraction in the base.

Definition 3.1. *The* quasiconformal distortion *of a cocycle $\mathcal{A}$ is the function*

$$Q_\mathcal{A}(x, n) = \|\mathcal{A}_x^n\| \cdot \|(\mathcal{A}_x^n)^{-1}\| = \frac{\sup\{\,\|\mathcal{A}_x^n(v)\| : \|v\| = 1\,\}}{\inf\{\,\|\mathcal{A}_x^n(v)\| : \|v\| = 1\,\}}, \quad x \in X \text{ and } n \in \mathbb{Z}.$$

The cocycle is uniformly quasiconformal *if $Q_\mathcal{A}(x, n) \le K$ for all x and n, and it is* conformal *if $Q_\mathcal{A}(x, n) = 1$ for all x and n.*

In the finite dimensional case, $\mathcal{A}_x^n$ maps the unit sphere to an ellipsoid, and $Q_\mathcal{A}(x, n)$ is the ratio of its largest and smallest semi-axes.

Definition 3.2. *A β-Hölder cocycle $\mathcal{A}$ over a hyperbolic system (X, f) is* fiber bunched *if there exist numbers $\theta < 1$ and L such that for all $x \in X$ and $n \in \mathbb{N}$,*

$$Q_\mathcal{A}(x, n) \cdot (\nu_x^n)^\beta < L\,\theta^n \quad \text{and} \quad Q_\mathcal{A}(x, -n) \cdot (\hat{\nu}_x^{-n})^\beta < L\,\theta^n,$$

where ν and $\hat{\nu}$ are as in (2.1), $\nu_x^n = \nu(f^{n-1}x) \cdots \nu(x)$ and $\hat{\nu}_x^{-n} = (\hat{\nu}_{f^{-n}x}^n)^{-1}$.

Clearly, conformal and uniformly quasiconformal cocycles are fiber bunched, and so are cocycles that are sufficiently close to conformal.

The limits of products $(\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n$ are useful in the study of non-commutative cocycles. These limits are called *holonomies*, and we will define them in Proposition 3.3 below. In the context of linear cocycles, this notion was introduced in [BV04, V08] and holonomies were further studied in [ASV13, KS13, S15, KS16]. Existence of holonomies was proved in [V08, ASV13] under the stronger "one-step" fiber bunching condition that there is $\theta < 1$ such that

(3.1) $$\|\mathcal{A}_x\| \cdot \|(\mathcal{A}_x)^{-1}\| \cdot \nu(x)^\beta < \theta \quad \text{for all } x \in X.$$

The result below, which also holds in the bundle setting [KS13], gives existence of holonomies under the weakest fiber bunching assumption given in Definition 3.2.

Proposition 3.3. *Let $\mathcal{A}$ be a β-Hölder linear or Banach cocycle over (X, f). If $\mathcal{A}$ is fiber bunched, then for every $x \in X$ and $y \in W^s(x)$ the limit*

(3.2) $$H_{x,y}^{\mathcal{A},s} = \lim_{n \to \infty} (\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n,$$

exists and satisfies

(H1) $H_{x,y}^{\mathcal{A},s} \in GL(V)$ *is an invertible linear map from* $\mathcal{E}_x$ *to* $\mathcal{E}_y$;

(H2) $H_{x,x}^{\mathcal{A},s} = Id$ *and* $H_{y,z}^{\mathcal{A},s} \circ H_{x,y}^{\mathcal{A},s} = H_{x,z}^{\mathcal{A},s}$, *which implies* $(H_{x,y}^{\mathcal{A},s})^{-1} = H_{y,x}^{\mathcal{A},s}$;

(H3) $H_{x,y}^{\mathcal{A},s} = (\mathcal{A}_y^n)^{-1} \circ H_{f^n x, f^n y}^{\mathcal{A},s} \circ \mathcal{A}_x^n$ *for all* $n \in \mathbb{N}$;

(H4) $\|H_{x,y}^{\mathcal{A},s} - Id\| \leq c\, dist(x,y)^\beta$, *where* c *is independent of* x *and* $y \in W_{loc}^s(x)$,
    and also $\|(\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n - Id\| \leq c\, dist(x,y)^\beta$ *for every* $n \in \mathbb{N}$.

The continuous map $H^{\mathcal{A},s} : (x,y) \mapsto H_{x,y}^{\mathcal{A},s}$, where $x \in X$, $y \in W^s(x)$, is called the *stable holonomy* for $\mathcal{A}$. The unstable holonomy $H^{\mathcal{A},u}$ for $\mathcal{A}$ is defined similarly:

$$H_{x,y}^{\mathcal{A},u} = \lim_{n \to \infty} \left((\mathcal{A}_y^{-n})^{-1} \circ (\mathcal{A}_x^{-n})\right) = \lim_{n \to \infty} \left(\mathcal{A}_{f^{-n}y}^n \circ (\mathcal{A}_{f^{-n}x}^n)^{-1}\right), \quad \text{where } y \in W^u(x).$$

Sometimes, any continuous map $H^{\mathcal{A},s} : (x,y) \mapsto H_{x,y}^{\mathcal{A},s}$ satisfying (H1-H3) is called a stable holonomy [ASV13], and the one in (3.2) is referred to as the *standard holonomy* to distinguish it [KS16]. It was proved in [KS13] that properties (H1-H4) uniquely define the standard holonomy. However, other maps may exist satisfying (H1-H3) with Hölder exponent lower than β in (H4), as discussed in [KS16] after Corollary 4.9.

Proof. We give a proof under the "one-step" fiber bunching assumption (3.1).

Let $x \in X$ and $y \in W_{\text{loc}}^s(x)$. The key step is to show that the sequence $((\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n)$ is Cauchy. Denoting $x_i = f^i(x)$ and $y_i = f^i(y)$, we obtain

$$(\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n = (\mathcal{A}_y^{n-1})^{-1} \circ \left((\mathcal{A}_{y_{n-1}})^{-1} \circ \mathcal{A}_{x_{n-1}}\right) \circ \mathcal{A}_x^{n-1} =$$

$$= (\mathcal{A}_y^{n-1})^{-1} \circ (\text{Id} + r_{n-1}) \circ \mathcal{A}_x^{n-1} = (\mathcal{A}_y^{n-1})^{-1} \circ \mathcal{A}_x^{n-1} + (\mathcal{A}_y^{n-1})^{-1} \circ r_{n-1} \circ \mathcal{A}_x^{n-1} =$$

$$= \cdots = \text{Id} + \sum_{i=0}^{n-1} (\mathcal{A}_y^i)^{-1} \circ r_i \circ \mathcal{A}_x^i, \quad \text{where } r_i = (\mathcal{A}_{y_i})^{-1} \circ \mathcal{A}_{x_i} - \text{Id}.$$

Since $y \in W_{\text{loc}}^s(x)$, for each $i \in \mathbb{N}$ we can estimate

$$\text{dist}(x_i, y_i) \leq \text{dist}(x,y) \cdot \nu_y^i = \text{dist}(x,y) \cdot \nu(y_0)\nu(y_1) \cdots \nu(y_{i-1}).$$

Since $\mathcal{A}$ is Hölder continuous with exponent β, we have

$$\|r_i\| \leq \|(\mathcal{A}_{y_i})^{-1}\| \cdot \|\mathcal{A}_{x_i} - \mathcal{A}_{y_i}\| \leq c_1 \text{dist}(x_i, y_i)^\beta \leq c_1 (\text{dist}(x,y)\, \nu_y^i)^\beta$$

Also,

$$\frac{\|\mathcal{A}_{x_k}\|}{\|\mathcal{A}_{y_k}\|} \leq 1 + \frac{\|\mathcal{A}_{x_k} - \mathcal{A}_{y_k}\|}{\|\mathcal{A}_{y_k}\|} \leq 1 + c_2 (\text{dist}(x_k, y_k))^\beta,$$

and we estimate

$$\|(\mathcal{A}_y^i)^{-1}\| \cdot \|\mathcal{A}_x^i\| \leq \|(\mathcal{A}_y)^{-1}\| \cdot \|(\mathcal{A}_{y_1})^{-1}\| \cdots \|(\mathcal{A}_{y_{i-1}})^{-1}\| \cdot \|\mathcal{A}_x\| \cdot \|\mathcal{A}_{x_1}\| \cdots \|\mathcal{A}_{x_{i-1}}\|$$

$$\leq \prod_{k=0}^{i-1} \|\mathcal{A}_{y_k}\| \|(\mathcal{A}_{y_k})^{-1}\| \cdot \prod_{k=0}^{i-1} \frac{\|\mathcal{A}_{x_k}\|}{\|\mathcal{A}_{y_k}\|} < \prod_{k=0}^{i-1} \theta\, \nu(y_k)^{-\beta} \cdot \prod_{k=0}^{i-1} \left(1 + c_2 (\text{dist}(x_k, y_k))^\beta\right).$$

Since the distance between x_k and y_k decreases exponentially, the second product is uniformly bounded and we obtain $\|(\mathcal{A}_y^i)^{-1}\| \cdot \|\mathcal{A}_x^i\| \leq c_3\, \theta^i\, (\nu_y^i)^{-\beta}$.

It follows that for every $i \geq 0$,

$$\|(\mathcal{A}_y^i)^{-1} \circ r_i \circ \mathcal{A}_x^i\| \leq \|(\mathcal{A}_y^i)^{-1}\| \cdot \|\mathcal{A}_x^i\| \cdot \|r_i\| \leq$$
$$\leq c_3\,\theta^i\,(\nu_y^i)^{-\beta} \cdot c_1(\mathrm{dist}(x,y)\,\nu_y^i)^\beta = c_4\,\mathrm{dist}(x,y)^\beta\,\theta^i.$$

Therefore, for every $n \in \mathbb{N}$,

$$\|\mathrm{Id} - (\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n\| \leq \sum_{i=0}^{n-1} \|(\mathcal{A}_y^i)^{-1} \circ r_i \circ \mathcal{A}_x^i\| \leq c_4\,\mathrm{dist}(x,y)^\beta \sum_{i=0}^{n-1}\theta^i \leq c\,\mathrm{dist}(x,y)^\beta.$$

Also, since

$$\|(\mathcal{A}_y^{n+1})^{-1} \circ \mathcal{A}_x^{n+1} - (\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n\| = \|(\mathcal{A}_y^n)^{-1} \circ r_n \circ \mathcal{A}_x^n\| \leq c_4\,\mathrm{dist}(x,y)^\beta\,\theta^n,$$

the sequence $\{(\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n\}$ is Cauchy and hence it has a limit $H_{xy}^s \in GL(V)$. Since the convergence is uniform on the set of pairs (x,y) where $y \in W_{\mathrm{loc}}^s(x)$, the map H^s is continuous. It is easy to see that the maps H_{xy}^s satisfy (H2, H3, H4). The map H_{xy}^s can be extended to any $y \in W^s(x)$ using (iii).

Existence of holonomies under the weaker fiber bunching of Definition 3.2 can be deduced by considering iterates of the cocycle and using uniqueness of holonomies, see [S15, Proposition 4.4] for details. □

4. Cohomology of group-valued cocycles over hyperbolic systems

4.1. **Introduction and main questions.** Cohomology of Hölder continuous cocycles over hyperbolic systems have been extensively studied starting with the seminal work of A. Livšic [Li71, Li72], where he established the following result. Parts (i) and (ii) are referred to as Periodic Point and Measurable Livšic Theorems, respectively.

Theorem 4.1 (Livšic Theorem). *Let $f : X \to X$ be a transitive Anosov diffeomorphism, and let $\alpha : X \to \mathbb{R}$ be a Hölder continuous function. Then*

(i) *If $\sum_{i=0}^{n-1} \alpha(f^i(p)) = 0$ whenever $f^n(p) = p$, then there exists a Hölder continuous function $\varphi : X \to \mathbb{R}$ such that $\alpha(x) = \varphi(f(x)) - \varphi(x)$ for all $x \in X$.*

(ii) *Let μ be an ergodic invariant measure with full support and local product structure. If there exists a μ-measurable function $\tilde{\varphi}$ such that $\alpha(x) = \tilde{\varphi}(f(x)) - \tilde{\varphi}(x)$ for μ-a.e. $x \in X$, then there exists a Hölder continuous function $\varphi : X \to \mathbb{R}$ such that $\alpha(x) = \varphi(f(x)) - \varphi(x)$ for all $x \in X$ and $\tilde{\varphi}(x) = \varphi(x)$ μ-a.e.*

The conclusions are that α is a *coboundary*, more specifically, it is Hölder continuously cohomologous to the zero cocycle. The notion of *cohomology* can be considered in various categories. For cocycles with values in a group G it is defined as follows.

Definition 4.2. *Cocycles $\mathcal{A}$ and $\mathcal{B}$ are (measurably, continuously) cohomologous if there exists a (measurable, continuous) function $C : X \to G$ such that*

$$\mathcal{A}_x^n = C(f^n x) \circ \mathcal{B}_x^n \circ C(x)^{-1} \quad \text{for all } x \in X \text{ and } n \in \mathbb{Z}.$$

The function C is called a conjugacy *between $\mathcal{A}$ and $\mathcal{B}$, or a* transfer map.

In addition to the case of $G = \mathbb{R}$, Livšic also established a result similar to Theorem 4.1 for Abelian G. Following his work, the questions of cohomology to the identity cocycle have been considered for non-Abelian groups, where they proved to be much more difficult.

For a G-valued cocycle $\mathcal{A}$, we call its *periodic data* the set of its products along the periodic orbits in the base, that is

$$\mathcal{A}_P = \{\mathcal{A}_p^n : \; p = f^n p, \; n \in \mathbb{N}\}.$$

A number of results extending Theorem 4.1 (i) were obtained [Li72, NT95, PWa01, LWi10] under additional assumptions on the group G or on the growth of the cocycle, such as compactness of the group or fiber bunching type conditions for the cocycle, see [KtN] for more details and references. Without such assumptions, the results were established in the following cases.

Theorem 4.3. *Let $\mathcal{A}$ be a Hölder continuous cocycle over (X, f) with values in one of the following groups*

$GL(d, \mathbb{R})$, *and more generally Lie groups and their closed subgroups (Kalinin* [K11]*);*
The group of invertible elements of a Banach algebra (Grabarnik, Gyusinsky [GrGu17]*);*
A group of diffeomorphisms of a compact manifold (Avila, Kocsard, Liu [AKL18]*).*

If $\mathcal{A}$ has identity periodic data, that is $\mathcal{A}_P = \{e_G\}$, then $\mathcal{A}$ is Hölder cohomologous to the identity cocycle.

Extending Theorem 4.1 (ii), Livšic established Hölder continuity of a measurable conjugacy to the identity cocycle for compact groups and groups with a bi-invariant metric [Li72]. Further results for Lie groups were obtained under additional assumptions such as boundedness of conjugacy or bunching of the cocycle [PaP97, NiP99, PWa01], and for $GL(d, \mathbb{R})$ without extra assumptions by Butler [Bt18].

Now we consider cohomology of two arbitrary Hölder continuous G-valued cocycles $\mathcal{A}$ and $\mathcal{B}$ over a hyperbolic system. This general problem is the most relevant for applications in the study of hyperbolic systems, as the derivative cocycle or its restrictions to the stable and unstable sub-bundles are not cohomologous to the identity cocycles. If G is an Abelian group, one can consider the cocycle $\mathcal{A} \circ \mathcal{B}^{-1}$ and thus reduce the problem to the case when $\mathcal{B}$ is the identity cocycle, i.e. $\mathcal{B}_x = e_G$, which was resolved by Livšic. For non-Abelian G, however, the general problem does not reduce to the special case $\mathcal{B} = e_G$ and is much more difficult. Moreover, the answers to some natural questions are negative in general. The main focus of the research has been on the following generalizations of the Livšic theorem.

Question 4.4. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ have conjugate periodic data, that is, whenever $p = f^n p$, $\mathcal{A}_p^n = C(p) \circ \mathcal{B}_p^n \circ C^{-1}(p)$ for some $C(p) \in G$. Does it follow that $\mathcal{A}$ and $\mathcal{B}$ are continuously cohomologous?*

Clearly, continuous cohomology implies conjugacy of the periodic data. As the answer to Question 4.4 is negative in general, one may ask:

Question 4.5. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ have equal periodic data, that is, $\mathcal{A}_p^n = \mathcal{B}_p^n$ whenever $f^n p = p$. Does it follow that $\mathcal{A}$ and $\mathcal{B}$ are continuously cohomologous?*

Finally, the following question extends the second part of Livšic theorem.

Question 4.6. *Let C be a measurable conjugacy between $\mathcal{A}$ and $\mathcal{B}$. Is C continuous? More precisely, does it coincide with a continuous conjugacy almost everywhere?*

As in the Livšic theorem, measurability should be understood with respect to an ergodic invariant measure with full support and local product structure, for example the measure of maximal entropy or the invariant volume.

Positive answers for Questions 4.6 and 4.5, as well as some results for cocycles with conjugate data, were obtained by Parry and Pollicott [PaP97, Pa99] for compact G and, somewhat more generally, by Schmidt [Schm99] for cocycles with "bounded distortion". First results outside this setting, including the following proposition, were obtained in [S13] for certain types of $GL(2, \mathbb{R})$-valued cocycles.

Proposition 4.7. [S13] *Let $f : X \to X$ be a transitive Anosov diffeomorphism, and let $\mathcal{A}$ and $\mathcal{B}$ be cocycles over (X, f) given by*

$$\mathcal{A}_x = k(x) \begin{bmatrix} 1 & \alpha(x) \\ 0 & 1 \end{bmatrix} \quad and \quad \mathcal{B}_x = l(x) \begin{bmatrix} 1 & \beta(x) \\ 0 & 1 \end{bmatrix},$$

where $k(x), l(x) \neq 0$ for all x, and α, β are not cohomologous to 0.

  (i) *If $\mathcal{A}$ and $\mathcal{B}$ have conjugate periodic data with $C(p)$ continuous at a periodic point p_0, then $\mathcal{A}$ and $\mathcal{B}$ are Hölder continuously cohomologous.*
  (ii) *Any measurable conjugacy between $\mathcal{A}$ and $\mathcal{B}$ is Hölder continuous.*
  (iii) *$\mathcal{A}$ and $\mathcal{B}$ are (measurably or Hölder) cohomologous if and only if there exist Hölder functions $\varphi(x)$ and $s(x)$ and a constant $c \neq 0$ such that*
  $k(x)/l(x) = \varphi(fx)/\varphi(x)$ and $\alpha(x) - c\beta(x) = s(fx) - s(x)$ for all $x \in X$.

We note that the cocycles in the proposition are fiber bunched, and that statements (i) and (ii) hold more generally for $GL(2, \mathbb{R})$-valued cocycles with one exponent at each periodic point.

General results on cohomolgy of fiber bunched cocycles were obtained in [S15] and [S17] for linear and Banach cocycles, respectively.

4.2. **Continuous conjugacy from periodic data.** As the following example demonstrates, the answer to Question 4.4 is negative in general, even if the cocycles are fiber bunched and $C(p)$ is bounded.

Example 4.8. [S13] *There exist cocycles $\mathcal{A}$ and $\mathcal{B}$ with generators $\mathcal{A}_x = \begin{bmatrix} 1 & \alpha(x) \\ 0 & 1 \end{bmatrix}$ and $\mathcal{B}_x = \begin{bmatrix} 1 & \beta(x) \\ 0 & 1 \end{bmatrix}$ arbitrarily close to the identity, such that $\mathcal{A}$ and $\mathcal{B}$ have conjugate periodic data with $C(p)$ uniformly bounded, but are not measurably cohomologous.*

The construction. By Proposition 4.7, $\mathcal{A}$ and $\mathcal{B}$ are measurably cohomologous only if there exist $c \neq 0$ and a Hölder function $s(x)$ such that $\alpha(x) - c\beta(x) = s(fx) - s(x)$. By Livšic theorem, the latter is equivalent to the fact that $\alpha(p, n) - c\beta(p, n) = 0$ for every periodic point $p = f^n p$, where $\alpha(p, n) = \alpha(p) + \alpha(fp) + \cdots + \alpha(f^{n-1}p)$.

Let $\varepsilon > 0$ and let α and β be Hölder continuous functions such that for two periodic points $p_1 = f^{n_1}(p_1)$ and $p_2 = f^{n_2}(p_2)$

$$\alpha(f^i p_1) = \beta(f^i p_1), \;\; 0 \leq i \leq n_1 - 1, \quad \alpha(f^i p_2) = 2\beta(f^i p_2), \;\; 0 \leq i \leq n_2 - 1,$$

and $0 < \beta(x) \leq \alpha(x) \leq 2\beta(x) < \varepsilon$ for all x. The function β can be chosen constant.

As $\alpha(p_1, n_1) = \beta(p_1, n_1)$ and $\alpha(p_2, n_2) = 2\beta(p_2, n_2)$, there is no constant c such that $\alpha(p, n) - c\beta(p, n) = 0$ for every periodic p.

Since $\alpha(p, n) > 0$ and $\beta(p, n) > 0$ at every periodic point $p = f^n p$, the functions α and β are not cohomologous to 0, and the matrices $\mathcal{A}_p^n$ and $\mathcal{B}_p^n$ are conjugate by the matrix $C(p) = \text{diag}\,(\alpha(p, n)/\beta(p, n), \, 1)$. Since $1 \leq \alpha(p, n)/\beta(p, n) \leq 2$ for every p, $C(p)$ is uniformly bounded. $\qquad\square$

Thus one has to make an assumption on $C(p)$ stronger than boundedness. If $C(p)$ is Hölder continuous, extending it to a function C on X and conjugating $\mathcal{B}$ by this extension reduces Question 4.4 to Question 4.5. We obtain a positive answer to Question 4.4 under a weaker assumption that $C(p)$ is Hölder continuous at one periodic point.

Assumptions. In Theorem 4.9, Corollary 4.10, Proposition 4.11 and Corollary 4.12, $\mathcal{A}$ and $\mathcal{B}$ are β-Hölder $GL(d, \mathbb{R})$ or $GL(V)$-valued cocycles over a hyperbolic system.

We note that in the theorem below fiber bunching is assumed only for $\mathcal{A}$, since fiber bunching for $\mathcal{B}$ can be obtained from the periodic data. This is discussed in Section 5.5.

Theorem 4.9. [S15, S17] *Suppose that $\mathcal{A}$ is fiber bunched, and $\mathcal{B}$ has conjugate periodic data such that for some periodic point p_0, $d(C(p), C(p_0)) \leq c\, dist(p, p_0)^\beta$ for every periodic point p. Then $\mathcal{A}$ and $\mathcal{B}$ are β-Hölder cohomologous. More precisely, there exists a unique β-Hölder continuous conjugacy $\bar{C}$ between $\mathcal{A}$ and $\mathcal{B}$ such that $\bar{C}(p_0) = C(p_0)$. Moreover, if $\mathcal{A}$ and $\mathcal{B}$ take values in a closed subgroup G_0 of $GL(d, \mathbb{R})$ or $GL(V)$ and $C(p_0) \in G_0$ then $\bar{C}(x) \in G_0$ for all x.*

The conjugacy $\bar{C}(p)$ does not necessarily coincide with $C(p)$ for $p \neq p_0$. For example, let $\mathcal{B}$ be the identity cocycle and let $\mathcal{A}_x = \bar{C}(fx) \circ \mathcal{B}_x \circ \bar{C}(x)^{-1} = \bar{C}(fx) \circ \bar{C}(x)^{-1}$, where $\bar{C} : X \to GL(d, \mathbb{R})$ is a Hölder continuous function with $\bar{C}(p_0) = \text{Id}$. Then $\mathcal{A}_p^n = \mathcal{B}_p^n = \text{Id}$ whenever $p = f^n p$, and thus we can take $C(p) = \text{Id}$ for each p.

As a corollary, we obtain a complete positive answer to Question 4.5 when one of the cocycles is fiber bunched. For $G = GL(d, \mathbb{R})$, this result was independently obtained by Backes [Bac15] under a stronger, uniform, version of bunching for both cocycles.

Corollary 4.10. [S15, S17] *If a cocycle $\mathcal{A}$ is fiber bunched and $\mathcal{B}_p^n = \mathcal{A}_p^n$ whenever $f^n p = p$, then $\mathcal{A}$ and $\mathcal{B}$ are β-Hölder cohomologous.*

Our approach to the proof is inspired by [Pa99, Schm99, PWa01]. An important role is played by the following observation.

Proposition 4.11. [S15] *Let $\mathcal{A}$ and $\mathcal{B}$ be two fiber bunched cocycles and let C be a β-Hölder continuous conjugacy between $\mathcal{A}$ and $\mathcal{B}$. Then C intertwines the holonomies for $\mathcal{A}$ and $\mathcal{B}$, that is,*

$$H_{x,y}^{\mathcal{A},s/u} = C(y) \circ H_{x,y}^{\mathcal{B},s/u} \circ C(x)^{-1} \quad \text{for every } x \in X \text{ and } y \in W^{s/u}(x).$$

We note, that this result holds for C with the same Hölder exponent β as the one used in the fiber bunching condition. However, there are examples of fiber bunched cocycles and conjugacy C that has lower Hölder exponent and does not intertwine their holonomies [KS16].

Proof. Let $x \in X$ and $y \in W^s(x)$. By iterating x and y forward the problem reduces to the case of $y \in W_{\text{loc}}^s(x)$. Since $\mathcal{A}(x) = C(fx) \circ \mathcal{B}_x \circ C(x)^{-1}$, we have

$$(\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n = C(y) \circ (\mathcal{B}_y^n)^{-1} \circ C(f^n y)^{-1} \circ C(f^n x) \circ \mathcal{B}_x^n \circ C(x)^{-1} =$$

(4.1) $$= C(y) \circ (\mathcal{B}_y^n)^{-1} \circ (\text{Id} + r_n) \circ \mathcal{B}_x^n \circ C(x)^{-1} =$$

$$= C(y) \circ (\mathcal{B}_y^n)^{-1} \circ \mathcal{B}_x^n \circ C(x)^{-1} + C(y) \circ (\mathcal{B}_y^n)^{-1} \circ r_n \circ \mathcal{B}_x^n \circ C(x)^{-1}.$$

Since C is Hölder continuous and $y \in W_{\text{loc}}^s(x)$, we can estimate

$$\|r_n\| = \|C(f^n y)^{-1} \circ C(f^n x) - \text{Id}\| \leq \|C(f^n y)^{-1}\| \cdot \|C(f^n x) - C(f^n y)\| \leq$$

$$\leq c_1 \, \text{dist}(f^n x, f^n y)^{\beta} \leq c_1 \, (\nu_y^n)^{\beta}.$$

Properties (H3) and (H4) of holonomies imply that there is a constant c_2 such that

$$\|\mathcal{B}_x^n\| \leq c_2 \|\mathcal{B}_y^n\| \quad \text{for all } x \in X, \, y \in W_{\text{loc}}^s(x), \text{ and } n \in \mathbb{N}.$$

Using estimates above and fiber bunching of the cocycle $\mathcal{B}$, we obtain

$$\|(\mathcal{B}_y^n)^{-1} \circ r_n \circ \mathcal{B}_x^n\| \leq \|(\mathcal{B}_y^n)^{-1}\| \cdot \|r_n\| \cdot c_3 \|\mathcal{B}_y^n\| \leq$$

$$\leq c_4 \|(\mathcal{B}_y^n)^{-1}\| \cdot \|\mathcal{B}_y^n\| \cdot (\nu_y^n)^{\beta} \leq c_5 \theta^n \to 0 \quad \text{as } n \to \infty.$$

Therefore the last term in (4.1) tends to 0.

Since $\lim_{n \to \infty} (\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n = H_{x,y}^{\mathcal{A},s}$ and $\lim_{n \to \infty} (\mathcal{B}_y^n)^{-1} \circ \mathcal{B}_x^n = H_{x,y}^{\mathcal{B},s}$, passing to the limit in (4.1) we obtain intertwining of the holonomies. $\qquad\square$

Let C be as in the proposition and let $C(x_0)$ be given for some $x_0 \in X$. It follows that for any $y \in W^s(x_0)$,

(4.2) $$C(y) = H_{y,x}^{\mathcal{B},s} \circ C(x_0) \circ H_{x,y}^{\mathcal{A},s}.$$

Since the stable manifold $W^s(x_0)$ is dense in X and C is Hölder continuous, such C is uniquely determined by its value at x_0.

Outline of the proof of Theorem 4.9. Since $\mathcal{A}$ is fiber bunched and $\mathcal{B}$ has conjugate periodic data with bounded $C(p)$, the cocycle $\mathcal{B}$ is also fiber bunched by Proposition 5.11. Therefore both cocycles have holonomies.

First, let us consider the case when the point p_0 is fixed. We conjugate the cocycle $\mathcal{B}$ by $C(p_0)$ and thus reduce the question to that of two cocycles with the same value

at p_0 and $C(p_0) = \text{Id}$. We still denote them $\mathcal{A}$ and $\mathcal{B}$. Following (4.2) we construct conjugacies between $\mathcal{A}$ and $\mathcal{B}$ along the stable and unstable manifolds of p_0 as

$$C^s(x) = H_{p_0,x}^{\mathcal{A},s} \circ C(p_0) \circ H_{x,p_0}^{\mathcal{B},s} = H_{p_0,x}^{\mathcal{A},s} \circ H_{x,p_0}^{\mathcal{B},s} \quad \text{for } x \in W^s(p_0),$$

$$C^u(x) = H_{p_0,x}^{\mathcal{A},u} \circ C(p_0) \circ H_{x,p_0}^{\mathcal{B},u} = H_{p_0,x}^{\mathcal{A},u} \circ H_{x,p_0}^{\mathcal{B},u} \quad \text{for } x \in W^u(p_0).$$

It is easy to verify that they are indeed conjugacies on the corresponding manifolds. The main part of the proof is to show that if x is a homoclinic point for p_0, that is, $x \in W^s(p_0) \cap W^u(p_0)$, then

$$(4.3) \qquad H_{x,p_0}^{\mathcal{A},s} \circ H_{p_0,x}^{\mathcal{A},u} = H_{x,p_0}^{\mathcal{B},s} \circ H_{p_0,x}^{\mathcal{B},u}, \quad \text{that is } C^s(x) = C^u(x) \overset{def}{=} \bar{C}(x).$$

To do this, we consider the orbit segment $\{f^{-m}x, \ldots, x, \ldots f^n(x)\}$ with sufficiently large m and n. Since both $f^{-m}x$ and $f^n(x)$ are close to p_0, by Anosov Closing Lemma 2.1 there exists a periodic point q close to p_0 whose iterates follow this orbit segment. Since q is close to p_0, $C(q)$ is Hölder close to $C(p_0) = \text{Id}$. Using the assumption that $C(q)$ conjugates the periodic values of $\mathcal{A}$ and $\mathcal{B}$ at q and carefully estimating the orbit products one can establish (4.3) in the limit as $m, n \to \infty$.

After (4.3) is proven, it is not difficult to show that the function $\bar{C}$ is β-Hölder continuous on the set of homoclinic points, and hence it can be extended to a Hölder continuous function on X. It is clear from the construction that if $\mathcal{A}(x), \mathcal{B}(x) \in G_0$ for all x and $C(p_0) \in G_0$, then the holonomies and $\bar{C}$ also take values in G_0.

In the case of a periodic point $p_0 = f^m(p_0)$, the above argument gives a conjugacy $\bar{C}$ between the cocycles $\mathcal{A}^m$ and $\mathcal{B}^m$ over the map f^m. Then an additional argument shows that $\bar{C}$ is also a conjugacy between $\mathcal{A}$ and $\mathcal{B}$. $\square$

We also obtained a result for a constant linear cocycle and its perturbation *without the fiber bunching assumption.* It is useful in the study of rigidity of Anosov automorphisms.

Corollary 4.12. [S15] *Let $\mathcal{A}$ be a constant $GL(d, \mathbb{R})$-valued cocycle, and let $\mathcal{B}$ be a Hölder continuous cocycle sufficiently close to $\mathcal{A}$. If $\mathcal{A}$ and $\mathcal{B}$ have conjugate periodic data and $C(p)$ is Hölder continuous at a fixed or periodic point p_0, then they are Hölder continuously cohomologous.*

Outline of the proof. Let $A(x) = A$ be the generator of $\mathcal{A}$. Let $\rho_1 < \cdots < \rho_\ell$ be the distinct moduli of the eigenvalues of A and let $\mathbb{R}^d = E_1^A \oplus \cdots \oplus E_\ell^A$ be the corresponding invariant splitting into the direct sum of the generalized eigenspaces. We denote $A_i = A|E_i^A$. It follows that for any $\varepsilon > 0$ there exists C_ε such that

$$C_\varepsilon^{-1}(\rho_i - \varepsilon)^n \le \|A_i^n u\| \le C_\varepsilon(\rho_i + \varepsilon)^n \quad \text{for any unit vector } u \in E_i^A,$$

and hence the cocycle $\mathcal{A}_i$ generated by A_i is fiber bunched for any $\beta > 0$. Moreover, any cocycle $\mathcal{B}$ with generator B sufficiently C^0 close to A has the corresponding invariant splitting $\mathbb{R}^d = E_1^B(x) \oplus \cdots \oplus E_\ell^B(x)$, which is close to that of $\mathcal{A}$ and is β-Hölder for some $\beta > 0$. The corresponding restrictions $\mathcal{B}_i$ satisfy similar estimates and hence are also fiber bunched. Since the conjugacy $C(p)$ maps $E_i^A(p)$ to $E_i^B(p)$, the cocycles $\mathcal{A}_i$ and $\mathcal{B}_i$ have conjugate periodic data. Hence by Theorem 4.9 they are conjugate via a Hölder

continuous function C_i, and we obtain a conjugacy between $\mathcal{A}$ and $\mathcal{B}$ as the direct sum of C_i. □

4.3. **Hölder continuity of a measurable conjugacy.** Fiber bunching of two cocycles does not guarantee continuity of a measurable conjugacy between them. An example of $GL(2, \mathbb{R})$-valued cocycles which are measurably but not continuously cohomologous was constructed in [PWa01]. In the example, both generators can be made arbitrarily close to the identity, and hence the cocycles are fiber bunched.

Example 4.13. [PWa01] *Arbitrarily close to the identity, there exist smooth functions*
$A(x) = \begin{bmatrix} \alpha(x) & \beta \\ 0 & 1 \end{bmatrix}$ *and* $B(x) = \begin{bmatrix} \alpha(x) & 0 \\ 0 & 1 \end{bmatrix}$ *such that the corresponding cocycles $\mathcal{A}$ and $\mathcal{B}$ are measurably, but not continuously cohomologous.*

Outline of the construction. Let f be an Anosov automorphism, and let x_0 be a fixed point. Let $\alpha(x)$ be a smooth function such that $\alpha(x_0) = 1$ and $1 - \varepsilon < \alpha(x) < 1$ for all $x \neq x_0$, and let $\beta = \varepsilon$. Since the matrices $A(x_0)$ and $B(x_0)$ are not conjugate, the cocycles $\mathcal{A}$ and $\mathcal{B}$ are not continuously cohomologous.

A measurable conjugacy is constructed in the form $C(x) = \begin{bmatrix} 1 & c(x) \\ 0 & 1 \end{bmatrix}$. Then

$A(x) = C(fx)B(x)C(x)^{-1}$ is equivalent to $\begin{bmatrix} \alpha(x) & \beta \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha(x) & c(fx) - \alpha(x)c(x) \\ 0 & 1 \end{bmatrix}$.

A function c such that $c(fx) = \beta + \alpha(x)c(x)$ is obtained as a series. Let

$$c_m(x) = \beta \cdot \left(1 + \alpha(f^{-1}x) + \alpha(f^{-1}x)\alpha(f^{-2}x) + \cdots + \alpha(f^{-1}x)\cdots\alpha(f^{-m}x)\right).$$

By the Birkhoff Ergodic Theorem, $(\alpha(f^{-1}x)\cdots\alpha(f^{-m}(x))^{1/m} \to \alpha < 1$ a.e. It follows that the sequence $\{c_m(x)\}$ converges to a limit $c(x)$ a.e., and the function c is measurable as a limit of continuous functions. The functions c_m satisfy the equation $c_m(fx) = \beta + \alpha(x)c_{m-1}(x)$, and passing to the limit it follows that $c(fx) = \beta + \alpha(x)c(x)$. □

Since the answer to Question 4.6 for fiber bunched cocycles is negative in general, we make a stronger assumption that one of the cocycles is uniformly quasiconformal.

Theorem 4.14. [S15] *Let $\mathcal{A}$ and $\mathcal{B}$ be linear cocycles, and let μ be an ergodic f-invariant measure on X with full support and local product structure. Suppose that $\mathcal{A}$ is fiber bunched and $\mathcal{B}$ is uniformly quasiconformal. Then any μ-measurable conjugacy between $\mathcal{A}$ and $\mathcal{B}$ is β-Hölder continuous, that is, it coincides with a β-Hölder continuous conjugacy on a set of full measure.*

A measure has local product structure if it is locally equivalent to the product of its conditional measures on the local stable and unstable manifolds. Examples of ergodic measures with full support and local product structure include the measure of maximal entropy, more generally Gibbs (equilibrium) measures of Hölder continuous potentials, and the invariant volume if it exists [PWa01].

Outline of the proof of Theorem 4.14. Let C be a μ-measurable conjugacy between $\mathcal{A}$ and $\mathcal{B}$. The key step in the proof is to show that C intertwines holonomies of $\mathcal{A}$ and $\mathcal{B}$ on a set of full measure, more precisely, there exists a set $Y \subset X$, $\mu(Y) = 1$, such that

$$(4.4) \qquad H_{x,y}^{\mathcal{A},s} = C(y) \circ H_{x,y}^{\mathcal{B},s} \circ C(x)^{-1} \quad \text{for all } x, y \in Y \text{ such that } y \in W^s(x).$$

This yields $C(y) = H_{x,y}^{\mathcal{A},s} \circ C(x) \circ H_{x,y}^{\mathcal{B},s}$ and then Hölder continuity of the holonomies easily implies essential Hölder continuity of C along W_{loc}^s, that is,

$$d(C(x), C(y)) \leq c \operatorname{dist}(x,y)^\beta \quad \text{for all } x, y \in Y \text{ such that } y \in W_{\mathrm{loc}}^s(x),$$

where c does not depend on x and y. Similarly one obtains essential Hölder continuity of C along W_{loc}^u. Then global Hölder continuity of C is obtained using the local product structure of stable and unstable manifolds and the local product structure of the measure.

To establish (4.4) we consider $x \in X$ and $y \in W_{\mathrm{loc}}^s(x)$ and, as in the proof of Proposition 4.11, we obtain

$$(4.5) \quad (\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n = C(y) \circ (\mathcal{B}_y^n)^{-1} \circ \mathcal{B}_x^n \circ C(x)^{-1} + C(y) \circ (\mathcal{B}_y^n)^{-1} \circ r_n \circ \mathcal{B}_x^n \circ C(x)^{-1},$$

where

$$\|r_n\| \leq \|C(f^n y)^{-1}\| \cdot \|C(f^n x) - C(f^n y)\|.$$

Since C is μ-measurable, by Lusin's theorem there exists a compact set $S \subset X$ with $\mu(S) > 1/2$ such that C is uniformly continuous on S and hence $\|C\|$ and $\|C^{-1}\|$ are bounded on S. Let Y be the set of points in X for which the frequency of visiting S equals $\mu(S) > 1/2$. By Birkhoff Ergodic Theorem $\mu(Y) = 1$. If x and y are in Y, there exists a sequence $\{n_i\}$ such that $f^{n_i} x$ and $f^{n_i} y$ are in S for all i. It follows that $\|r_{n_i}\| \to 0$ as $i \to \infty$. The product

$$\|(\mathcal{B}_y^n)^{-1}\| \cdot \|\mathcal{B}_x^n\| \leq \|H_{f^n x, f^n y}^{\mathcal{B},s}\| \cdot \|(\mathcal{B}_x^n)^{-1}\| \cdot \|H_{x,y}^{\mathcal{B},s}\| \cdot \|\mathcal{B}_x^n\|$$

is uniformly bounded since the cocycle $\mathcal{B}$ is uniformly quasiconformal. Thus for every x and y in Y such that $y \in W_{\mathrm{loc}}^s(x)$, the second term in (4.5) tends to 0 along a subsequence, and (4.4) follows. $\square$

The next result extends Theorem 4.14 to the infinite dimensional setting. One of the difficulties here is that the space $(GL(V), d)$ is not separable even if V is. To use the tools of the theory of measurable functions, such as Lusin's theorem, we work with the strong operator topology, i.e., the topology of pointwise convergence. We assume that $\mathcal{B}$ takes values in a precompact set, which for a finite dimensional V is equivalent to uniform boundedness of $\mathcal{B}$ in $(GL(V), d)$.

Theorem 4.15. [S17] *Let $\mathcal{A}$ and $\mathcal{B}$ be $GL(V)$-valued cocycles. Suppose that the Banach space V is separable, $\mathcal{A}$ is fiber bunched and $\mathcal{B}$ takes values in a subset of $GL(V)$ that is precompact in the strong operator topology. Let μ be an ergodic invariant measure with full support and local product structure. Then any μ-measurable conjugacy between $\mathcal{A}$ and $\mathcal{B}$ coincides with a β-Hölder continuous conjugacy on a set of full measure.*

The μ-measurability of the conjugacy means that the preimage of each Borel set in $GL(V)$ is μ-measurable and the conjugacy equation holds μ-almost everywhere. We note that Borel σ-algebra in $GL(V)$ is the same for the metric d and for the strong operator topology.

5. Periodic approximation of Lyapunov exponents and growth estimates for cocycles

5.1. **Definitions and results.** First we consider a β-Hölder continuous linear cocycle $\mathcal{A}$ on a finite-dimensional vector bundle $\mathcal{E}$ over a hyperbolic system (X, f). We discuss Lyapunov exponents of $\mathcal{A}$ with respect to an f-invariant ergodic Borel probability measure μ on X. The vector bundle $\mathcal{E}$ is trivial on a set of full measure and hence $\mathcal{A}$ can be viewed as a $GL(d, \mathbb{R})$-valued cocycle on a set of full measure. Continuity of the cocycle implies that the integrability assumption in the theorem below is satisfied, and so we obtain the Lyapunov exponents of $\mathcal{A}$ and the corresponding Lyapunov decomposition of $\mathcal{E}$ on a set of full measure.

Theorem 5.1 (Oseledets Multiplicative Ergodic Theorem (MET)). [O68]
Let f be an invertible ergodic measure-preserving transformation of a Lebesgue probability space (X, μ). Let $\mathcal{A}$ be a measurable $GL(d, \mathbb{R})$-valued cocycle over f satisfying $\log \|\mathcal{A}_x\| \in L^1(X, \mu)$ and $\log \|\mathcal{A}_x^{-1}\| \in L^1(X, \mu)$.

Then there exist numbers $\lambda_1 < \cdots < \lambda_\ell$, an f-invariant set Λ with $\mu(\Lambda) = 1$, and an $\mathcal{A}$-invariant Lyapunov decomposition $\mathcal{E}_x = \mathcal{E}_x^1 \oplus \cdots \oplus \mathcal{E}_x^\ell$ for $x \in \Lambda$ such that

(i) $\lim_{n \to \pm\infty} \frac{1}{n} \log \|\mathcal{A}_x^n v\| = \lambda_i$ *for any $i = 1, \ldots, \ell$ and any $0 \neq v \in \mathcal{E}_x^i$, and*

(ii) $\lim_{n \to \pm\infty} \frac{1}{n} \log |\det \mathcal{A}_x^n| = \sum_{i=1}^\ell m_i \lambda_i$, *where $m_i = \dim \mathcal{E}_x^i$.*

The numbers $\lambda_1, \ldots, \lambda_\ell$ are called the *Lyapunov exponents* of $\mathcal{A}$ with respect to μ. The exponents and the decomposition depend on the choice of μ.

By Lyapunov exponents of $\mathcal{A}$ at a periodic point p we mean the Lyapunov exponents of $\mathcal{A}$ with respect to the invariant measure μ_p on the orbit of p. They equal $(1/k)$ of the logarithms of the absolute values of the eigenvalues of $\mathcal{A}_p^k$, where k is a period of p. The following result on periodic approximation of Lyapunov exponents for finite-dimensional $\mathcal{E}$ was established by Kalinin.

Theorem 5.2. [K11] *Let (X, f) be a hyperbolic dynamical system, let $\mathcal{A}$ be a Hölder continuous linear cocycle over f, and let μ be an ergodic invariant measure for f.*

Then the Lyapunov exponents $\lambda_1 \leq \cdots \leq \lambda_d$ of $\mathcal{A}$ with respect to μ, listed with multiplicities, can be approximated by the Lyapunov exponents of $\mathcal{A}$ at periodic points. More precisely, for any $\varepsilon > 0$ there exists a periodic point $p \in X$ for which the Lyapunov exponents $\lambda_1^{(p)} \leq \cdots \leq \lambda_d^{(p)}$ of $\mathcal{A}$ satisfy $|\lambda_i - \lambda_i^{(p)}| < \varepsilon$ for $i = 1, \ldots, d$.

For $GL(V)$-valued cocycles with infinite-dimensional V, there is no Oseledets MET in general, but **upper and lower Lyapunov exponents** $\lambda(\mathcal{A}, \mu)$ and $\chi(\mathcal{A}, \mu)$ of $\mathcal{A}$

with respect to μ can be defined as follows.

(5.1)
$$\lambda(\mathcal{A}, \mu) = \lim_{n \to \infty} \frac{1}{n} \log \|\mathcal{A}_x^n\| \quad \text{for } \mu\text{-a.e. } x \in X,$$

$$\chi(\mathcal{A}, \mu) = \lim_{n \to \infty} \frac{1}{n} \log \|(\mathcal{A}_x^n)^{-1}\|^{-1} \quad \text{for } \mu\text{-a.e. } x \in X.$$

These limits exist and are constant μ almost everywhere. Indeed, the sequence of functions $a_n(x) = \frac{1}{n} \log \|\mathcal{A}_x^n\|$ satisfies

(5.2) $a_{n+m}(x) \le a_m(x) + a_n(f^m x) \quad$ for all $x \in X$ and $n, m \in \mathbb{N}$,

that is, $\{a_n(x)\}$ is a *subadditive cocycle*. By the Subadditive Ergodic Theorem for μ-a.e. $x \in X$

$$\lim_{n \to \infty} \frac{a_n(x)}{n} = \lim_{n \to \infty} \frac{a_n(\mu)}{n} = \inf\left\{ \frac{a_n(\mu)}{n} \ : \ n \in \mathbb{N} \right\}, \quad \text{where } a_n(\mu) = \int_X a_n(x) \, d\mu.$$

This limit is called the *exponent* of the subadditive cocycle. A similar argument applies to the second limit in (5.1)

If V is finite-dimensional, then $\lambda(\mathcal{A}, \mu)$ and $\chi(\mathcal{A}, \mu)$ are precisely the largest and smallest of the Lyapunov exponents given by the Oseledets MET, that is, $\lambda(\mathcal{A}, \mu) = \lambda_\ell$ and $\chi(\mathcal{A}, \mu) = \lambda_1$.

In the infinite-dimensional setting, it is not always possible to approximate $\lambda(\mathcal{A}, \mu)$ and $\chi(\mathcal{A}, \mu)$ by Lyapunov exponents at periodic points. The following proposition is based on an example by Gurvits of a pair of operators whose joint spectral radius is greater than the generalized spectral radius [Gu95].

Proposition 5.3. [KS17] *There exists a locally constant cocycle $\mathcal{A}$ over a full shift on two symbols and an ergodic invariant measure μ such that $\lambda(\mathcal{A}, \mu) > \sup_{\mu_p} \lambda(\mathcal{A}, \mu_p)$, where the supremum is taken over all uniform measures μ_p on periodic orbits.*

Nonetheless, the upper and lower exponents can be approximated in terms of norms at periodic points as follows.

Theorem 5.4. [KS17] *Let (X, f) be a hyperbolic system, let μ be an ergodic f-invariant measure on X, and let $\mathcal{A}$ be a Hölder continuous $GL(V)$-valued cocycle over f.*

Then for any $\varepsilon > 0$ there exists a periodic point $p = f^k p$ such that

$$\left| \lambda(\mathcal{A}, \mu) - \frac{1}{k} \log \|\mathcal{A}_p^k\| \right| < \varepsilon \quad \text{and} \quad \left| \chi(\mathcal{A}, \mu) - \frac{1}{k} \log \|(\mathcal{A}_p^k)^{-1}\|^{-1} \right| < \varepsilon.$$

Moreover, for any $N \in \mathbb{N}$ there exists such $p = f^k p$ with $k > N$.

We note that for the measures μ_p on the orbit of $p = f^k p$,

(5.3)
$$\lambda(\mathcal{A}, \mu_p) = \lim_{m \to \infty} \frac{1}{mk} \log \|(\mathcal{A}_p^k)^m\| = \frac{1}{k} \log \lim_{m \to \infty} \|(\mathcal{A}_p^k)^m\|^{1/m} =$$
$$= \frac{1}{k} \log (\text{spectral radius of } \mathcal{A}_p^k) \le \frac{1}{k} \log \|\mathcal{A}_p^k\|,$$

and the inequality can be strict, even in the finite-dimensional case. Also, $\chi(\mathcal{A}, \mu_p) \geq \frac{1}{k} \log \|(\mathcal{A}_p^k)^{-1}\|^{-1}$. Thus for a point p as in the theorem we have one sided estimates

$$\lambda(\mathcal{A}, \mu_p) < \lambda(\mathcal{A}, \mu) + \varepsilon \quad \text{and} \quad \chi(\mathcal{A}, \mu_p) > \chi(\mathcal{A}, \mu) - \varepsilon.$$

Remark 5.5. *Theorems 5.2 and 5.4 can be strengthened to conclude the existence of a periodic point $p = f^k p$ which gives simultaneous approximation for finitely many cocycles $\mathcal{A}_{(i)}$, $i = 1, \ldots, m$, over f with values in $GL(d_i, \mathbb{R})$ and $GL(V_i)$, respectively.*

In [KS18] we extended the periodic approximation results in [WaSu10, K11, KS17] to non-uniformly hyperbolic setting. We recall that a measure is *hyperbolic* if all its Lyapunov exponents for the derivative cocycle Df are nonzero, and a periodic point p is *hyperbolic* if the corresponding measure μ_p is hyperbolic.

Theorem 5.6. [KS18] *Let f be a $C^{1+H\ddot{o}lder}$ diffeomorphism of a compact manifold X, let μ be a hyperbolic ergodic f-invariant measure on X, and let $\mathcal{A}$ be a $GL(V)$-valued Hölder continuous cocycle over f.*

(i) *If $V = \mathbb{R}^d$, then the Lyapunov exponents $\lambda_1 \leq \cdots \leq \lambda_d$ of $\mathcal{A}$ with respect to μ, listed with multiplicities, can be approximated by the Lyapunov exponents of $\mathcal{A}$ at a hyperbolic periodic point p as in Theorem 5.2.*

(ii) *For any Banach space V, the upper and lower Lyapunov exponents λ and χ of $\mathcal{A}$ with respect to μ can be approximated as in Theorem 5.4, where $p = f^k p$ is a hyperbolic periodic point.*

5.2. **Outline of the proof of Theorem 5.4.** We consider a full measure set $\Lambda = \Lambda_\mu$ of points $x \in X$ such that

$$\lim_{n \to \infty} \frac{1}{n} \log \|\mathcal{A}_{f^{-n}x}^n\| = \lambda = \lim_{n \to \infty} \frac{1}{n} \log \|\mathcal{A}_x^n\| \quad \text{and}$$

$$\lim_{n \to \infty} \frac{1}{n} \log \|\mathcal{A}_x^{-n}\| = -\chi = \lim_{n \to \infty} \frac{1}{n} \log \|(\mathcal{A}_x^n)^{-1}\|.$$

We construct a certain version of Lyapunov, or adapted, norm which allows us to control the norms of $\mathcal{A}_x$ and $(\mathcal{A}_x)^{-1}$.

Let γ be as in the Anosov Closing Lemma 2.1 and β be the Hölder exponent of $\mathcal{A}$. We fix ε such that $0 < \varepsilon < \varepsilon_0 = \beta\gamma/4$. For a point $x \in \Lambda$ we define the ε-*Lyapunov norm* $\|.\|_x = \|.\|_{x,\varepsilon}$ on V_x as follows. For $u \in V_x$,

$$(5.4) \qquad \|u\|_x = \|u\|_{x,\varepsilon} = \sum_{n=0}^{\infty} \|\mathcal{A}_x^n(u)\| \, e^{-(\lambda+\varepsilon)n} + \sum_{n=1}^{\infty} \|\mathcal{A}_x^{-n}(u)\| \, e^{(\chi-\varepsilon)n}.$$

By the definition of Λ, both series converge exponentially.

For any points $x, y \in \Lambda$ and any linear map $A : V_x \to V_y$ we denote its operator norm with respect to the Lyapunov norms by

$$\|A\|_{y \leftarrow x} = \sup \left\{ \|A(u)\|_{y,\varepsilon} : u \in V_x, \ \|u\|_{x,\varepsilon} = 1 \right\}$$

and estimate it using (5.4)

$$(5.5) \qquad \|\mathcal{A}_x\|_{fx \leftarrow x} \leq e^{\lambda+\varepsilon} \quad \text{and} \quad \|\mathcal{A}_x^{-1}\|_{f^{-1}x \leftarrow x} \leq e^{-\chi+\varepsilon}.$$

Since the Lyapunov norm depends only measurably on x, we provide a comparison with the standard norm. In [KS17, Proposition 3.1] we showed that there exists an f-invariant set $\mathcal{R} \subset \Lambda$ with $\mu(\mathcal{R}) = 1$ and a measurable function $K = K_\varepsilon$ such that for all $x \in \mathcal{R}$

$$(5.6) \qquad \|u\| \leq \|u\|_{x,\varepsilon} \leq K(x)\|u\| \quad \text{for all } u \in \mathcal{E}_x, \quad \text{and}$$

$$(5.7) \qquad K(x)e^{-\varepsilon|n|} \leq K(f^n x) \leq K(x)e^{\varepsilon|n|} \quad \text{for all } n \in \mathbb{Z}.$$

Also, for any points $x, y \in \mathcal{R}$ the inequality (5.6) yields

$$(5.8) \qquad \|A\|_{y \leftarrow x} \leq K(y)\|A\| \quad \text{and} \quad \|A\| \leq K(x)\|A\|_{y \leftarrow x}.$$

We let $\mathcal{R}_t = \{x \in \mathcal{R} : K(x) \leq t\}$. Then $\mu(\mathcal{R}_t) \to 1$ as $t \to \infty$.

An important role in the arguments is played by the following estimate of $\|\mathcal{A}_y^n\|$ along an orbit close to a regular one. We recall that $\varepsilon < \beta\gamma$, where γ is as in Lemma 2.1.

Lemma 5.7. *There exists a constant $c = c(\mathcal{A}, \beta\gamma - \varepsilon)$ such that for any point x in $\mathcal{R}_t$ with $f^m x$ in $\mathcal{R}_t$ and any point $y \in X$ such that the orbit segments $x, fx, \ldots, f^m x$ and $y, fy, \ldots, f^m y$ satisfy with some $\delta > 0$*

$$(5.9) \qquad dist(f^i x, f^i y) \leq \delta e^{-\gamma \min\{i, m-i\}} \quad \text{for every } i = 0, \ldots, m$$

we have for all $0 \leq n \leq m$

$$(5.10) \qquad \|\mathcal{A}_y^n\| \leq t\, \|\mathcal{A}_y^n\|_{f^n x \leftarrow x} \leq t\, e^{ct\delta^\beta} e^{n(\lambda+\varepsilon)}.$$

A similar estimate can be obtained for $\|(\mathcal{A}_y^n)^{-1}\|$. We note that y may not be a regular point, and so we consider the norm of the linear map $\mathcal{A}_y^n$ with respect to the Lyapunov norms at x and $f^n x$.

Proof. We denote $x_i = f^i x$ and $y_i = f^i y$, $i = 0, \ldots, m$, and use (5.5) to estimate the Lyapunov norm for $0 < n \leq m$

$$(5.11)$$
$$\|\mathcal{A}_y^n\|_{x_n \leftarrow x_0} \leq \prod_{i=0}^{n-1} \|\mathcal{A}_{y_i}\|_{x_{i+1} \leftarrow x_i} \leq \prod_{i=0}^{n-1} \|\mathcal{A}_{x_i}\|_{x_{i+1} \leftarrow x_i} \cdot \|(\mathcal{A}_{x_i})^{-1} \circ \mathcal{A}_{y_i}\|_{x_i \leftarrow x_i}$$

$$\leq e^{n(\lambda+\varepsilon)} \prod_{i=0}^{n-1} \|(\mathcal{A}_{x_i})^{-1} \circ \mathcal{A}_{y_i}\|_{x_i \leftarrow x_i} = e^{n(\lambda+\varepsilon)} \prod_{i=0}^{n-1} \|\operatorname{Id} + \Delta_i\|_{x_i \leftarrow x_i}.$$

Since $\mathcal{A}_x$ is β-Hölder and $\|(\mathcal{A}_x)^{-1}\|$ is uniformly bounded,

$$\|\Delta_i\| \leq \|(\mathcal{A}_{x_i})^{-1}\| \cdot \|\mathcal{A}_{y_i} - \mathcal{A}_{x_i}\| \leq M'\text{dist}(x_i, y_i)^\beta \leq M' \left(\delta e^{-\gamma \min\{i, m-i\}}\right)^\beta,$$

where the constant M' depends only on the cocycle $\mathcal{A}$. Since both x and $f^m x$ are in $\mathcal{R}_t$ we have $K(x_i) \leq te^{\varepsilon \min\{i, m-i\}}$ by (5.7), so using the first part of (5.8) we conclude that

$$\|\Delta_i\|_{x_i \leftarrow x_i} \leq K(x_i)\|\Delta_i\| \leq te^{\varepsilon \min\{i, m-i\}} M'\delta^\beta e^{-\gamma\beta \min\{i, m-i\}} = M't\, \delta^\beta\, e^{(\varepsilon-\beta\gamma) \min\{i, m-i\}}.$$

Taking logarithm in (5.11) and using the above inequality we obtain

$$\log \|\mathcal{A}_y^n\|_{x_n \leftarrow x_0} - n(\lambda + \varepsilon) \leq \sum_{i=0}^{n-1} \log \left(\| \operatorname{Id} + \Delta_i \|_{x_i \leftarrow x_i} \right) \leq \sum_{i=0}^{n-1} \log \left(1 + \| \Delta_i \|_{x_i \leftarrow x_i} \right)$$

$$\leq \sum_{i=0}^{n-1} \| \Delta_i \|_{x_i \leftarrow x_i} \leq M' t \delta^\beta \sum_{i=0}^{n-1} e^{(\varepsilon - \alpha\gamma) \min\{i, m-i\}} \leq M' t \delta^\beta \cdot 2 \sum_{i=0}^{\infty} e^{(\varepsilon - \beta\gamma)i} = c \, t \delta^\beta$$

since $\varepsilon < \beta\gamma$. The constant c depends only on $\mathcal{A}$ and $(\beta\gamma - \varepsilon)$. We conclude that

(5.12) $$\|\mathcal{A}_y^n\|_{x_n \leftarrow x_0} \leq e^{c \, t \delta^\beta} e^{n(\lambda + \varepsilon)}.$$

Since $K(x_0) \leq t$, using the second part of (5.8) we can also estimate the standard norm

$$\|\mathcal{A}_y^n\| \leq K(x_0) \|\mathcal{A}_y^n\|_{x_n \leftarrow x_0} \leq t e^{c \, t \delta^\beta} e^{n(\lambda + \varepsilon)}.$$

$\square$

Now we can obtain a point $p = f^k p$ such that $\frac{1}{k} \log \|\mathcal{A}_p^k\| \leq \lambda(\mathcal{A}, \mu) + 2\varepsilon$ as follows. We take $\mathcal{R}_t$ with a sufficiently large t and $x \in \mathcal{R}_t$ so that $f^k x \in \mathcal{R}_t$ and is sufficiently close to x to apply Anosov Closing Lemma. Then there exists $p = f^k p$ satisfying (5.9). It follows from (5.10) that for a sufficiently large k we have

(5.13) $$\frac{1}{k} \log \|\mathcal{A}_p^k\| \leq \frac{1}{k} \log(t \, e^{c \, t \delta^\alpha}) + (\lambda + \varepsilon) < \lambda + 2\varepsilon.$$

Now we outline how to find a point $p = f^k p$ for which $\frac{1}{k} \log \|\mathcal{A}_p^k\|$ also satisfies a corresponding lower estimate. In addition to Lemma 5.7, this part of the argument uses an approach developed in [GrGu17] and Karlsson-Margulis Lemma [KaM99, Proposition 4.2]. The latter provides indices n for which a sub-additive cocycle behaves almost additively in a certain sense. For future reference, we formulate its generalization for the case of several cocycles [KS18].

Proposition 5.8. [KaM99, KS18] *Let* $a_n^{(1)}(x), \ \ldots, a_n^{(m)}(x)$ *be integrable subadditive cocycles with exponents* $\lambda^{(1)} > -\infty, \ \ldots, \ \lambda^{(m)} > -\infty$, *respectively, over an ergodic measure-preserving system* (X, f, μ). *Then there exists a set* $E \subset X$ *with* $\mu(E) = 1$ *such that for each* $x \in E$ *and each* $\varepsilon > 0$ *there exists an integer* $L = L(x, \varepsilon)$ *so that the set* $S = S(x, \varepsilon, L)$ *of integers* n *satisfying the following condition is infinite:*

(5.14) $$a_n^{(j)}(x) - a_{n-i}^{(j)}(f^i x) \geq (\lambda^{(j)} - \varepsilon)i \quad \text{for all } L \leq i \leq n \text{ and } 1 \leq j \leq m.$$

We apply this proposition to the sub-additive cocycle $a_n(x) = \log \|\mathcal{A}_x^n\|$, consider a typical point $x \in \mathcal{R}_t$, and take a sufficiently large $n \in S(x, \varepsilon, L)$. Then we have

(5.15) $$\|\mathcal{A}_x^n\| \cdot \|\mathcal{A}_{f^i x}^{n-i}\| \geq e^{(\lambda - \varepsilon)i} \quad \text{for all } i \text{ such that } L \leq i \leq n.$$

Now we would like to find k so that $f^k x \in \mathcal{R}_t$ and is sufficiently close to x to apply Anosov Closing Lemma, and also so that the ratio k/n is bounded, specifically,

$$n(1 + \sigma) \leq k \leq n(1 + 2\sigma), \quad \text{where } \sigma = 4\varepsilon/(\beta\gamma).$$

Existence of such k relies on [GrGu17, Lemma 8] or its refinement [KS18, Lemma 5.1].

The upper bound on k ensures that $\|\mathcal{A}_p^k\|$ and $\|\mathcal{A}_p^n\|$ are roughly comparable, so that it suffices to estimate $\|\mathcal{A}_p^n\|$ and show that it must grow as $\|\mathcal{A}_x^n\|$, with exponential rate at least $\lambda - \varepsilon$. For this it suffices to prove that $\|\mathcal{A}_x^n - \mathcal{A}_p^n\| \leq \frac{1}{2}\|\mathcal{A}_x^n\|$. Thus the main part of the argument is to estimate $\|\mathcal{A}_x^n - \mathcal{A}_p^n\|$. For this we write

$$\mathcal{A}_x^n - \mathcal{A}_p^n = \mathcal{A}_{x_1}^{n-1} \circ (\mathcal{A}_x - \mathcal{A}_p) + (\mathcal{A}_{x_1}^{n-1} - \mathcal{A}_{p_1}^{n-1}) \circ \mathcal{A}_p =$$

$$= \cdots = \sum_{i=0}^{n-1} \mathcal{A}_{x_{i+1}}^{n-(i+1)} \circ (\mathcal{A}_{x_i} - \mathcal{A}_{p_i}) \circ \mathcal{A}_p^i, \quad \text{and hence}$$

$$\|\mathcal{A}_x^n - \mathcal{A}_p^n\| \leq \sum_{i=0}^{n-1} \|\mathcal{A}_{x_{i+1}}^{n-(i+1)}\| \cdot \|\mathcal{A}_{x_i} - \mathcal{A}_{p_i}\| \cdot \|\mathcal{A}_p^i\|.$$

The third term in the product can be estimated by Lemma 5.7, $\|\mathcal{A}_p^i\| \leq t\, e^{ct\delta^\alpha} e^{i(\lambda+\varepsilon)}$. The second term is estimated using Hölder continuity of $\mathcal{A}$ and exponential closeness of the trajectories

$$\|\mathcal{A}_{x_i} - \mathcal{A}_{p_i}\| \leq c \operatorname{dist}(f^i x, f^i p)^\beta \leq c\, e^{-\gamma\beta \min\{i, k-i\}} \delta^\beta,$$

To estimate the first term, we use our choice of n and apply (5.15) to get

$$\|\mathcal{A}_{x_{i+1}}^{n-(i+1)}\| \leq \|\mathcal{A}_x^n\| \cdot e^{-(i+1)(\lambda-\varepsilon)} \quad \text{for all } L \leq i \leq n.$$

Combining these inequalities and estimating the first L terms separately, we obtain

$$\|\mathcal{A}_x^n - \mathcal{A}_p^n\| \leq c(\delta)\, \|\mathcal{A}_x^n\| \sum_{i=0}^{n-1} e^{2i\varepsilon} e^{-\gamma\beta \min\{i, k-i\}}, \quad \text{where } c(\delta) \to 0 \text{ as } \delta \to 0.$$

To complete the argument we need an estimate for the sum independent of k and n. This follows from the lower bound $k \geq (1+\sigma)n$, which ensures that

$$\beta\gamma \min\{i, k-i\} \geq 4\varepsilon i \quad \text{for } i = 0, \ldots, n.$$

This completes the outline of the estimate from below, $\frac{1}{k} \log \|\mathcal{A}_p^k\| \geq \lambda - c'\varepsilon$. The upper estimate (5.13) also holds for the same x and k since both $x, f^k x$ are in $\mathcal{R}_t$. This completes the approximation of the upper Lyapunov exponent λ in Theorem 5.4.

The arguments above allow us to obtain simultaneous approximation of the upper exponents for several cocycles $\mathcal{A}_{(1)}, \ldots, \mathcal{A}_{(m)}$ as in Remark 5.5. The only modification needed is to consider the intersection of the corresponding regular sets $\mathcal{R}_{t,(j)}$ for all $\mathcal{A}_{(j)}$ and use Proposition 5.8 for m subadditive cocycles $a^{(j)}(x) = \log \|\mathcal{A}_{(j)}\|$, $j = 1, \ldots, m$. The same argument with minor modifications can be also applied to the cocycle $\mathcal{A}^{-1}$ to obtain a simultaneous approximation of the lower exponent χ. $\qquad\square$

5.3. **Outline of the proof of Theorem 5.2.** In place of the Lyapunov norm given by (5.4), we use the norm generated by the usual Lyapunov scalar product $\langle \cdot, \cdot \rangle_{x,\varepsilon}$ defined as follows. For a fixed $\varepsilon > 0$ and for any x in the set Λ of the Lyapunov regular points from Theorem 5.1,

$\langle u, v \rangle_{x,\varepsilon} = 0$ if $u \in \mathcal{E}_x^i$, $v \in \mathcal{E}_x^j$, $i \neq j$, and

$\langle u, v \rangle_{x,\varepsilon} = d \sum_{n \in \mathbb{Z}} \langle \mathcal{A}_x^n u, \mathcal{A}_x^n v \rangle e^{-2\lambda_i n - \varepsilon |n|}$ if $u, v \in \mathcal{E}_x^i$, $i = 1, \dots, m$.

The properties of this norm are given in [BaPe, Sections 3.5.1-3.5.3].

We take $\mathcal{R}'_t = \{x \in \Lambda : K'(x) \leq t\}$, where and $K'(x)$ is a function similar to $K(x)$ which gives the estimate (5.6) for this Lyapunov norm.

First we approximate the largest exponent λ_ℓ. We again take $x \in \mathcal{R}'_t$ and k so that $f^k x \in \mathcal{R}'_t$, k and t are large enough, and $f^k x$ is sufficiently close to x. Then we obtain a periodic point $p = f^k p$ by Anosov Closing Lemma. As in the approximation of λ from above in Theorem 5.4, we obtain estimate (5.13) which, using (5.3), also implies the desired upper estimate

$$\lambda_\ell(\mathcal{A}, \mu_p) \leq \frac{1}{k} \log \|\mathcal{A}_p^k\| < \lambda_\ell + 2\varepsilon.$$

The lower estimate $\lambda_\ell(\mathcal{A}, \mu_p) > \lambda_\ell - 3\varepsilon$ is obtained by constructing a $\mathcal{A}_p^k$-invariant cone $K \subseteq \mathcal{E}_p$ where vectors are expanded by a factor at least $e^{k(\lambda_\ell - 3\varepsilon)}$, see [K11]. The cone K is defined as a cone of vectors whose direction is close, in the Lyapunov metric at x, to the Lyapunov subspace $\mathcal{E}_x^\ell$ corresponding to λ_ℓ. Then for any vector in K its forward Lyapunov exponent is at least $\lambda_\ell - 3\varepsilon$, and hence the same is true for $\lambda_\ell(\mathcal{A}, \mu_p)$.

To approximate all Lyapunov exponents of $\mathcal{A}$ we consider cocycles $\wedge^i \mathcal{A}$ induced by $\mathcal{A}$ on the i-fold exterior powers $\wedge^i \mathbb{R}^d$, for $i = 1, \dots, d$. The largest Lyapunov exponent of $\wedge^i \mathcal{A}$ is $(\lambda_d + \dots + \lambda_{d-i+1})$, where $\lambda_1 \leq \dots \leq \lambda_d$ are the Lyapunov exponents of $\mathcal{A}$ listed with multiplicities. If a periodic point $p = f^k p$ satisfies

$$|(\lambda_d + \dots + \lambda_{d-i+1}) - (\lambda_d^{(p)} + \dots + \lambda_{d-i+1}^{(p)})| \leq 3\varepsilon \quad \text{for } i = 1, \dots, d,$$

then we obtain the approximation $|\lambda_i - \lambda_i^{(p)}| \leq 3d\varepsilon$ for all $i = 1, \dots, d$, completing the proof of Theorem 5.2.

A similar argument shows that one can obtain a simultaneous approximation of all Lyapunov exponents for several cocycles. $\qquad \square$

5.4. **Growth estimates for cocycles.** As a corollary of Theorem 5.4, we obtain growth estimates for the norm and the quasiconformal distortion of a cocycle $\mathcal{A}$ in terms of the growth at periodic points.

Corollary 5.9. [S17] $\mathcal{A}$ be a Hölder continuous Banach cocycle over f. Then

(i) $\lim_{n \to \infty} \sup \left\{ \|\mathcal{A}_x^n\|^{1/n} : x \in X \right\} = \limsup_{k \to \infty} \sup \left\{ \|\mathcal{A}_p^k\|^{1/k} : p = f^k p \in X \right\}.$

   In particular, if for some numbers C and s we have $\|\mathcal{A}_p^k\| \leq C e^{sk}$ whenever $p = f^k p$, then for each $\varepsilon > 0$ there exists a number C_ε such that

$$\|\mathcal{A}_x^n\| \leq C_\varepsilon \, e^{(s+\varepsilon)n} \quad \text{for all } x \in X \text{ and } n \in \mathbb{N}.$$

(ii) $\lim_{n \to \pm\infty} \sup \left\{ Q_\mathcal{A}(x, n)^{1/|n|} : x \in X \right\} = \limsup_{k \to \infty} \sup \left\{ Q_\mathcal{A}(p, k)^{1/k} : p = f^k p \in X \right\}.$

   In particular, if for some numbers C and s we have $Q_\mathcal{A}(p, k) \leq C e^{sk}$ whenever

$p = f^k p$, *then for each $\varepsilon > 0$ there exists a number C'_ε such that*

$$Q_{\mathcal{A}}(x, n) \leq C'_\varepsilon \, e^{(s+\varepsilon)|n|} \quad \text{for all } x \in X \text{ and } n \in \mathbb{Z}.$$

The number

$$\hat{\lambda}(\mathcal{A}) = \limsup_{n \to \infty} \left\{ \frac{1}{n} \log \|\mathcal{A}^n_x\| : x \in X \right\}$$

gives the maximal exponential growth rate for a cocycle $\mathcal{A}$. Part (i) of Corollary 5.9 can also be deduced from [GrGu17], where the authors obtained a generalization of Livsic periodic point theorem for Banach cocycles, and in particular showed that for each $\varepsilon > 0$ there exists a periodic point $p = f^k p$ such that $\frac{1}{k} \log \|\mathcal{A}^k_p\| > \hat{\lambda}(\mathcal{A}) - \varepsilon$.

Outline of the proof of (ii). Let $\hat{\sigma}(\mathcal{A})$ and $\hat{\sigma}_P(\mathcal{A})$ be the logarithms of the left-hand side and the right-hand side of (ii), and let

$$q_n(x) = \log Q_{\mathcal{A}}(x, n) = \log \|\mathcal{A}^n_x\| + \log \|(\mathcal{A}^n_x)^{-1}\|.$$

The sequence $\{q_n(x)\}$ is a subadditive cocycle over f, and its exponent with respect to an ergodic measure μ is $\gamma(q, \mu) = \lambda(\mathcal{A}, \mu) - \chi(\mathcal{A}, \mu)$. The sequence of numbers $\hat{q}_n = \sup_{x \in X} q_n(x)$ is subadditive and hence $\hat{\gamma}(q) = \lim_{n \to \infty} (\hat{q}_n / n)$ exists. It follows from [Schr98] that $\hat{\gamma}(q) = \sup_\mu \gamma(q, \mu)$, where the supremum is taken over all ergodic f-invariant measures, see also [KS13, Proposition 4.9] for a direct proof of a similar result. Then using Theorem 5.4 we obtain

$$\limsup_{n \to \infty} \left\{ \log Q(x, n)^{1/n} : x \in X \right\} = \hat{\gamma}(q) = \sup_\mu \gamma(q, \mu) = \sup_\mu (\lambda(\mathcal{A}, \mu) - \chi(A, \mu))$$

$$\leq \sup \left\{ \frac{1}{k} \log \|\mathcal{A}^k_p\| + \frac{1}{k} \log \|(\mathcal{A}^k_p)^{-1}\| : p = f^k p, \ k \geq N \right\}$$

$$= \sup \left\{ \frac{1}{k} \log Q_{\mathcal{A}}(p, k) : p = f^k p, \ k \geq N \right\}.$$

It follows that $\hat{\sigma}(\mathcal{A}) \leq \hat{\sigma}_P(\mathcal{A})$, and the opposite inequality clearly holds.

Now, suppose that $Q(p, k) \leq C e^{sk}$ whenever $p = f^k p$. Then we have

$$s \geq \hat{\sigma}_P(\mathcal{A}) = \hat{\sigma}(\mathcal{A}) = \hat{\gamma}(q) = \lim_{n \to \infty} \hat{q}_n / n.$$

It follows that for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\hat{q}_n \leq (s + \varepsilon)n$ for all $n > N$ and hence $Q(x, n) \leq e^{(s+\varepsilon)n}$ for all $x \in X$ and $n > N$. Taking

$$C'_\varepsilon = \max \{ Q(x, n) : x \in X \text{ and } 1 \leq n \leq N \},$$

we obtain $Q(x, n) \leq C'_\varepsilon e^{(s+\varepsilon)n}$ for all $x \in X$ and $n \in \mathbb{N}$. The statements for negative n follows since $Q(x, -n) = Q(f^{-n} x, n)$ $\qquad\square$

In finite dimensional case, using Theorem 5.2, we can similarly obtain growth estimates using the Lyapunov exponents rather than norms.

Corollary 5.10. [K11, S15] $\mathcal{A}$ *be a Hölder continuous linear cocycle over (X, f).*

(i) *If for some number s we have $\lambda_\ell(\mathcal{A}, \mu_p) \leq s$ for every invariant measure μ_p on a periodic orbit, then for each $\varepsilon > 0$ there exists a number C_ε such that*

$$\|\mathcal{A}_x^n\| \leq C_\varepsilon \, e^{(s+\varepsilon)n} \quad \text{for all } x \in X \text{ and } n \in \mathbb{N}.$$

(ii) *If for some s we have $\lambda_\ell(\mathcal{A}, \mu_p) - \lambda_1(\mathcal{A}, \mu_p) \leq s$ for every invariant measure μ_p on a periodic orbit, then for each $\varepsilon > 0$ there exists a number C_ε' such that*

$$Q_{\mathcal{A}}(x, n) \leq C_\varepsilon' \, e^{(s+\varepsilon)|n|} \quad \text{for all } x \in X \text{ and } n \in \mathbb{Z}.$$

5.5. Obtaining fiber bunching from periodic data.

We recall that in Theorem 4.9 we assumed fiber bunching for only one of the cocycles as fiber bunching for the other one can be deduced from the conjugacy of their periodic data. Now we explain how this is done using the results on approximation of Lyapunov exponents.

Proposition 5.11. [S15, S17] *Let $\mathcal{A}$ and $\mathcal{B}$ be Hölder continuous cocycles over (X,f).*

(i) *If $\mathcal{A}$ and $\mathcal{B}$ take values in $GL(d, \mathbb{R})$, $\mathcal{A}$ is fiber bunched and $\mathcal{B}$ has conjugate periodic data, then $\mathcal{B}$ is also fiber bunched.*

(ii) *If $\mathcal{A}$ and $\mathcal{B}$ take values in $GL(V)$, $\mathcal{A}$ is fiber bunched and $\mathcal{B}$ has conjugate periodic data with bounded $C(p)$, then $\mathcal{B}$ is also fiber bunched.*

Part (ii) follows from the next result that establishes fiber bunching of a cocycle from fiber bunching of its periodic data. Part (i) follows from a similar characterization of fiber bunching in finite dimensional case using Lyapunov exponents at periodic orbits [S15, Corollary 4.2].

Proposition 5.12. [S17] *Let $\mathcal{B}$ be a β-Hölder cocycle over (X, f). Suppose that there exist numbers $\tilde{\theta} < 1$ and $\tilde{L}$ such that whenever $f^n p = p$,*

(5.16) $$Q_{\mathcal{B}}(p, n) \cdot (\nu_p^n)^\beta < \tilde{L} \, \tilde{\theta}^n \quad \text{and} \quad Q_{\mathcal{B}}(p, -n) \cdot (\hat{\nu}_p^{-n})^\beta < \tilde{L} \, \tilde{\theta}^n.$$

Then $\mathcal{B}$ is fiber bunched.

Outline of the proof. If functions $\nu(x) = \nu$ and $\hat{\nu}(x) = \hat{\nu}$ are constant, then the result follows immediately from Corollary 5.9 (ii). In general, one can apply an argument as in the proof of Corollary 5.9 to the sub-additive cocycle $\tilde{q}_n(x) = \log Q_{\mathcal{B}}(x, n) \cdot \log \nu_n(x)^\beta$ and use simultaneous approximation of exponents as in Remark 5.5. $\qquad\square$

6. Boundedness, conformality, and reductions

It is often useful to determine whether a cocycle is cohomologous to one with values in a smaller group, such as orthogonal or conformal, since cocycles with values in these groups are relatively well understood. Such reduction problems are highly nontrivial: Schmidt asked in [Schm99] whether a $GL(d, \mathbb{R})$-valued cocycle $\mathcal{A}$ with $\|\mathcal{A}_x^n\|$ uniformly bounded is Hölder cohomologous to an orthogonal one. In [KS10] we showed this assuming boundedness of the periodic data only. Moreover, we established a general criterium for a linear cocycle to be isometric or conformal in terms of its periodic data.

In this section we view a linear cocycle $\mathcal{A}$ as an automorphism of a vector bundle $\mathcal{E}$ over X with the fiber map $\mathcal{A}_x : \mathcal{E}_x \to \mathcal{E}_{fx}$, and we do not assume triviality of the bundle.

6.1. **Conformal structures.** A conformal structure on $\mathbb{R}^d$, $d \geq 2$, is a class of proportional inner products. Since an inner product identifies with a symmetric positive definite $d \times d$ matrix, we can uniquely represent a conformal structure by such a matrix with determinant 1. The group $GL(d, \mathbb{R})$ acts on $\mathcal{C}^d$ via

$$D(C) = (\det D^T D)^{-1/d} \cdot D^T C D, \quad \text{where } C \in \mathcal{C}^d \text{ and } D \in GL(d, \mathbb{R}).$$

Its subgroup $SL(d, \mathbb{R})$ acts transitively on $\mathcal{C}^d$ since every $C \in \mathcal{C}^d$ can be expressed as $C = D(\text{Id})$, where D is the positive square root of C. The stabilizer of the identity matrix in $SL(d, \mathbb{R})$ is $SO(d, \mathbb{R})$, and thus $\mathcal{C}^d$ can be identified with $SL(d, \mathbb{R})/SO(d, \mathbb{R})$. It is well-know that $\mathcal{C}^d = SL(d, \mathbb{R})/SO(d, \mathbb{R})$ is a Riemannian symmetric space with a certain $GL(d, \mathbb{R})$-invariant metric of non-positive curvature, see [T86, p.327].

For a vector bundle $\mathcal{E} \to X$ we can consider a bundle $\mathcal{C}$ over X whose fiber $\mathcal{C}_x$ is the space of conformal structures on $\mathcal{E}_x$. Using a background Riemannian metric on $\mathcal{E}$, the space $\mathcal{C}_x$ can be identified with the space of symmetric positive linear operators on $\mathcal{E}_x$ with determinant 1. We equip the fibers of $\mathcal{C}$ with the Riemannian metric as above. A continuous (measurable) section σ of $\mathcal{C}$ is called a continuous (measurable) *conformal structure* on $\mathcal{E}$. A measurable conformal structure σ is called *bounded* if the distance between $\sigma(x)$ and $\tau(x)$ is bounded on X for a continuous conformal structure τ on X.

An invertible linear map $A : \mathcal{E}_x \to \mathcal{E}_y$ induces an isometry from $\mathcal{C}_x$ to $\mathcal{C}_y$ via

$$A(C) = (\det(A^*A))^{1/d} \cdot (A^{-1})^* C A^{-1},$$

where $C \in \mathcal{C}_x$ is a conformal structure viewed as an operator, and A^* is the adjoint of A. If $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ is a linear cocycle over f, we say that a conformal structure σ on $\mathcal{E}$ is $\mathcal{A}$-*invariant* if

$$\mathcal{A}_x(\sigma(x)) = \sigma(fx) \quad \text{for all } x \in X.$$

6.2. **Conformality and uniform quasiconformality.**
Let $\mathcal{A}$ be a f linear cocycle over (X, f). We recall that its quasiconformal distortion is

$$Q_{\mathcal{A}}(x, n) = \|\mathcal{A}_x^n\| \cdot \|(\mathcal{A}_x^n)^{-1}\| = \frac{\max\{\|\mathcal{A}_x^n(v)\| : v \in \mathcal{E}_x, \|v\| = 1\}}{\min\{\|\mathcal{A}_x^n(v)\| : v \in \mathcal{E}_x, \|v\| = 1\}}, \quad x \in X, \, n \in \mathbb{Z}.$$

The cocycle is called *uniformly quasiconformal* if $Q_{\mathcal{A}}(x, n) \leq K$ for all for all x and n, and it is called *conformal* if $Q_{\mathcal{A}}(x, n) = 1$ for all x and n.

Clearly, $\mathcal{A}$ is conformal with respect to a Riemannian metric on $\mathcal{E}$ if and only if it preserves the conformal structure associated with this metric. We note that the notion of uniform quasiconformality does not depend on the choice of a continuous metric on $\mathcal{E}$. So if $\mathcal{A}$ preserves a continuous or bounded conformal structure on $\mathcal{E}$ then $\mathcal{A}$ is uniformly quasiconformal on $\mathcal{E}$ with respect to any continuous metric on $\mathcal{E}$. The theorem below shows that the converse is also true.

Theorem 6.1. [S02, KS10] *Let $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ be a Hölder continuous linear cocycle over a hyperbolic dynamical system (X, f). If $\mathcal{A}$ is uniformly quasiconformal then it preserves a Hölder continuous conformal structure on $\mathcal{E}$, equivalently, $\mathcal{A}$ is conformal with respect to a Hölder continuous Riemannian metric on $\mathcal{E}$.*

The first step of the proof is obtaining a bounded measurable conformal structure. We apply observations made by Sullivan and Tukia for quasiconformal group actions.

Proposition 6.2. *Let f be a homeomorphism of a compact metric space X and let $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ be a continuous linear cocycle over f. If $\mathcal{A}$ is uniformly quasiconformal then it preserves a bounded Borel measurable conformal structure σ on $\mathcal{E}$.*

Proof. Let τ be a continuous conformal structure on $\mathcal{E}$. We denote by $\tau(x)$ the conformal structure on $\mathcal{E}_x$, $x \in X$. We consider the set

$$S(x) = \{\, (\mathcal{A}_{f^n x}^{-n})(\tau(f^n x)) : \ n \in \mathbb{Z} \,\}$$

in $\mathcal{C}_x$, the space of conformal structures on $\mathcal{E}_x$. Since $\mathcal{A}$ is uniformly quasiconformal, the sets $S(x)$ have uniformly bounded diameters. Since the space $\mathcal{C}_x$ is simply connected and has non-positive curvature, every non-empty bounded set in $\mathcal{C}_x$ is contained in a unique closed ball of the smallest radius, see for example [T86, Lemma E]. The center of this ball is called a circumcenter of the set. For each $x \in X$ we denote by $\sigma(x)$ the circumcenter of $S(x)$.

It follows from the construction that the conformal structure σ is $\mathcal{A}$-invariant and its distance from τ is bounded. For any $k \geq 0$, the set

$$S_k(x) = \{\, (\mathcal{A}_{f^n x}^{-n})(\tau(f^n x)) : \ |n| \leq k \,\}$$

depends continuously on x in Hausdorff distance, and so does the center $\sigma_k(x)$ of the smallest ball containing $S_k(x)$. Since $S_k(x) \to S(x)$ as $k \to \infty$ for any x, the conformal structure σ is the pointwise limit of continuous conformal structures $\sigma_k(x)$. Hence σ is Borel measurable. $\qquad\square$

To complete the proof of Theorem 6.1 we use Theorem 6.3 below to obtain Hölder continuity of σ. The theorem applies since uniform quasiconformality of $\mathcal{A}$ implies its fiber bunching.

Theorem 6.3. [KS13] *Let (X, f) be a hyperbolic dynamical system, let $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ be a β-Hölder fiber bunched linear cocycle over f, and let μ be an ergodic f-invariant probability measure with full support and local product structure. Then any $\mathcal{A}$-invariant μ-measurable conformal structure on $\mathcal{E}$ is β-Hölder.*

Outline of the proof. We recall that a fiber bunched cocycle has stable and unstable holonomies, see Proposition 3.3. The main part of the proof is establishing essential invariance of the conformal structure under holonomies.

Lemma 6.4. *Let H^s be the stable holonomy for a linear cocycle $\mathcal{A}$. If τ is a μ-measurable $\mathcal{A}$-invariant conformal structure then τ is essentially H^s-invariant, i.e., there is a set $G \subset X$ of full measure such that $\tau(y) = H_{xy}^s(\tau(x))$ for all $x, y \in G$ such that $y \in W_{loc}^s(x)$.*

Proof. Let $x_i = f^i x$. Since $\mathcal{A}_y^n$ induces an isometry and τ is $\mathcal{A}$-invariant, we obtain

$$\text{dist}\,(\tau(y), H_{xy}^s(\tau(x))) = \text{dist}(\mathcal{A}_y^n(\tau(y)), \mathcal{A}_y^n H_{xy}^s(\tau(x))) =$$
$$= \text{dist}(\tau(y_n), H_{x_n y_n}^s \mathcal{A}_x^n(\tau(x))) = \text{dist}(\tau(y_n), H_{x_n y_n}^s(\tau(x_n))) \leq$$
$$\leq \text{dist}(\tau(y_n), \tau(x_n)) + \text{dist}(\tau(x_n), H_{x_n y_n}^s(\tau(x_n))).$$

Since τ is μ-measurable, by Lusin's Theorem there exists a compact set $S \subset X$ with $\mu(S) > 1/2$ on which τ is uniformly continuous and hence bounded. Let G be the set of points in X for which the frequency of visiting S equals $\mu(S) > 1/2$. By Birkhoff Ergodic Theorem, $\mu(G) = 1$.

Suppose that both x and y are in G. Then there exists a sequence $\{n_i\}$ such that both x_{n_i} and y_{n_i} are in S. Since $y \in W^s_{\mathrm{loc}}(x)$, $\mathrm{dist}(x_{n_i}, y_{n_i}) \to 0$ and hence $\mathrm{dist}(\tau(x_{n_i}), \tau(y_{n_i})) \to 0$ by uniform continuity of τ on S. Since $\|H^s_{x_{n_i} y_{n_i}} - \mathrm{Id}\| \to 0$ and τ is bounded on S we obtain

$$\mathrm{dist}(\tau(x_{n_i}), \, H^s_{x_{n_i} y_{n_i}}(\tau(x_{n_i}))) \to 0.$$

Here we used the fact that for any linear map X sufficiently close to the identity we have $\mathrm{dist}\,(\tau, X(\tau)) \le k(\tau) \cdot \|X - \mathrm{Id}\|$, where $k(\tau)$ is bounded on bounded sets in $\mathcal{C}^d$ [KS10]. We conclude that $\mathrm{dist}\,(\tau(y), H^s_{xy}(\tau(x))) = 0$ and thus τ is essentially H^s-invariant. $\qquad\square$

Similarly, τ is essentially H^u-invariant. By Hölder continuity of holonomies (H4) this implies essential Hölder continuity of τ along the stable and unstable leaves:

$$\mathrm{dist}\,(\tau(x), \tau(y)) \le C \,\mathrm{dist}(x, y)^\beta \ \ \text{if } y \in W^{s/u}_{\mathrm{loc}}(x) \text{ and both } x, y \in G.$$

To complete the proof of Theorem 6.3 we use the local product structure of μ and the local product structure of the stable and unstable foliations to show that τ coincides μ-a.e. with a Hölder continuous conformal structure on $\mathrm{supp}\,\mu = X$. $\qquad\square$

6.3. **Conformality and isometry from periodic data.**

If a cocycle $\mathcal{A}$ over an Anosov diffeomorphism f is conformal or uniformly quasiconformal then, clearly, there exists a constant C_{per} such that $Q_\mathcal{A}(p, n) \le C_{per}$ for every periodic point p and n such that $p = f^n p$. The next theorem shows that the converse is also true. A similar result was also obtained in [LWi10] under an extra bunching-type assumption.

Theorem 6.5. [KS10] *Let* $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ *be a Hölder continuous linear cocycle over a hyperbolic system* (X, f).

(i) *If there exists a constant* C_{per} *such that* $Q_\mathcal{A}(p, n) \le C_{per}$ *whenever* $f^n p = p$, *then* $\mathcal{A}$ *is conformal with respect to a Hölder continuous Riemannian metric on* $\mathcal{E}$.

(ii) *If* $\mathcal{A}$ *has bounded periodic data, that is, there exists a constant* C'_{per} *such that* $\max\,\{\|\mathcal{A}^n_p\|, \|(\mathcal{A}^n_p)^{-1}\|\} \le C'_{per}$ *whenever* $f^n p = p$, *then* $\mathcal{A}$ *is an isometry with respect to a Hölder continuous Riemannian metric on* $\mathcal{E}$.

Remark 6.6. *For a cocycle on a trivial bundle* $X \times \mathbb{R}^d$ *given by* $A : X \to GL(d, \mathbb{R})$ *the theorem implies Hölder cohomology to a cocycle with values in the conformal or orthogonal subgroup. Indeed, let* $C(x)$ *be the unique positive square root of the symmetric positive definite matrix that defines the Riemannian metric at* x. *Then* C *is a desired Hölder continuous conjugacy.*

Proof of Theorem 6.5.

(i) The assumption and Proposition 5.12 yield fiber bunching of $\mathcal{A}$. By Theorem 6.1 it suffices to show that $\mathcal{A}$ is uniformly quasiconformal on $\mathcal{E}$. Let $z \in X$ be a point with

dense orbit $\mathcal{O} = \{f^k z : k \in \mathbb{Z}\}$. We show that the quasiconformal distortion $Q_{\mathcal{A}}(z, k)$ is uniformly bounded in $k \in \mathbb{Z}$. Since $Q_{\mathcal{A}}(x, k)$ is continuous on X for each k, this implies uniform quasiconformality.

Let δ_0 be as in Anosov Closing Lemma, and let $0 < \delta < \delta_0$. We consider two points of $\mathcal{O}$ with $k_1 < k_2$ and $\mathrm{dist}(f^{k_1} z, f^{k_2} z) < \delta$, and we denote $x = f^{k_1} z$ and $n = k_2 - k_1$. Then there exists $p \in X$ with $f^n p = p$ such that $\mathrm{dist}(f^i x, f^i p) \le c\delta$ for $i = 0, \ldots, n$. Let $y \in W_{\mathrm{loc}}^s(p) \cap W_{\mathrm{loc}}^u(x)$. By Proposition 3.3 (H4),

$$\|(\mathcal{A}_p^n)^{-1} \circ \mathcal{A}_y^n - \mathrm{Id}\| \le c'\delta^\beta \quad \text{and} \quad \|(\mathcal{A}_y^{-n})^{-1} \circ \mathcal{A}_x^{-n} - \mathrm{Id}\| \le c'\delta^\beta.$$

It follows that if δ is sufficiently small, then

$$Q_{\mathcal{A}}(y, n)/Q_{\mathcal{A}}(p, n) \le (1 + c'\delta^\beta)/(1 - c'\delta^\beta) \le 2 \quad \text{and} \quad Q_{\mathcal{A}}(x, n)/Q_{\mathcal{A}}(y, n) \le 2.$$

Thus $Q_{\mathcal{A}}(x, n) \le 4\, Q_{\mathcal{A}}(p, n) \le 4 C_{per}$.

We take $m > 0$ such that the set $\{f^j z; \ |j| \le m\}$ is δ-dense in X, and set

$$Q_m = \max\{Q_{\mathcal{A}}(z, j) : \ |j| \le m\}.$$

Then for any $k > m$ there exists j, $|j| \le m$, such that $\mathrm{dist}(f^k z, f^j z) \le \delta$ and hence

$$Q_{\mathcal{A}}(z, k) \le Q_{\mathcal{A}}(z, j) \cdot Q_{\mathcal{A}}(f^j z, k - j) \le Q_m \cdot 4 C_{per}.$$

The case of $k < -m$ is considered similarly. Thus $Q_{\mathcal{A}}(z, k)$ is uniformly bounded.

(ii) By (i), the cocycle is conformal with respect to a Hölder continuous Riemannian metric g on $\mathcal{E}$. Hence there exists a positive Hölder continuous function $a(x)$ such that

$$\|\mathcal{A}_x(v)\|_{g(fx)} = a(x) \cdot \|v\|_{g(x)} \quad \text{for each } x \in X \text{ and each } v \in \mathcal{E}_x.$$

We seek a positive Hölder continuous function $\varphi(x)$ such that the renormalized metric $\tilde{g}(x) = g(x)/\varphi(x)$ is invariant, that is

$$\varphi(fx) \cdot \|\mathcal{A}_x(v)\|_{\tilde{g}(fx)} = a(x)\varphi(x) \cdot \|v\|_{\tilde{g}(x)}, \quad \text{equivalently,} \quad a(x) = \varphi(fx)/\varphi(x).$$

Let p be a periodic point of a period n. The boundedness assumption for *all* periods does not allow $a(p)a(fp)\cdots a(f^{n-1}p) = \|\mathcal{A}_p^n\|_{g(p)}$ to be greater or less than one. Hence $a(p)a(fp)\cdots a(f^{n-1}p) = 1$ whenever $p = f^n$, and by Livšic theorem the equation $a(x) = \varphi(fx)/\varphi(x)$ has a Hölder continuous solution. $\qquad \square$

Part (ii) can also be obtained without using (i) as follows. First, similarly to (i) we can show that boundedness of the periodic data implies boundedness of the cocycle, that is, of the set $\{\mathcal{A}_x^n : x \in X, \ n \in \mathbb{Z}\}$. Then an analog of Proposition 6.2 can be proven for inner products instead of conformal structures. This would give a bounded measurable invariant family of inner products. Its Hölder continuity can be established as in Theorem 6.3.

One may try to make weaker assumptions on the periodic data, for example that for each p, there is a uniform bound $C(p)$ on $Q_{\mathcal{A}}(p, n)$ for all periods n. This is equivalent to each of the following three statements:

$\mathcal{A}_p^n$ is diagonalizable over $\mathbb{C}$ with its eigenvalues equal in modulus;

$\mathcal{A}_p^n$ is conjugate to a conformal linear map;

There exists a $\mathcal{A}_p^n$-invariant conformal structure on $\mathcal{E}_p$.

In fact, the periodic assumption in the first part of Theorem 6.5 is equivalent to having such conformal structures for all periodic points uniformly bounded. Without such boundedness assumption, the theorem does not hold in dimension higher than two, as the following example demonstrates.

Proposition 6.7. [KS10] *Let (X, f) be a hyperbolic system and let $\mathcal{E} = X \times \mathbb{R}^d$, $d \geq 3$. For any $\varepsilon > 0$ there exists a Lipschitz continuous linear cocycle $\mathcal{A} : \mathcal{E} \to \mathcal{E}$, which its generator ε-close to the identity, such that for all periodic points $p \in X$ the return maps $\mathcal{A}_p^n : \mathcal{E}_p \to \mathcal{E}_p$ are conjugate to orthogonal maps, but $\mathcal{A}$ is not conformal with respect to any continuous Riemannian metric on $\mathcal{E}$.*

Outline of the construction. Let $\mathcal{E} = X \times \mathbb{R}^3$ and $\mathcal{A}_x = \begin{bmatrix} \cos \alpha(x) & -\sin \alpha(x) & \varepsilon \\ \sin \alpha(x) & \cos \alpha(x) & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Let S be a closed f-invariant set in X without periodic points and let $\alpha : X \to \mathbb{R}$ be a Lipschitz continuous function satisfying

$$\alpha(x) = 0 \text{ for } x \in S \text{ and } 0 < \alpha(x) \leq \varepsilon \text{ for } x \notin S$$

Then for $x \in S$, $\mathcal{A}_x^n = \begin{bmatrix} 1 & 0 & \mathbf{n}\varepsilon \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and so $Q_\mathcal{A}(x, n) \to \infty$ as $n \to \infty$.

Thus $\mathcal{A}$ cannot be conformal with respect to a continuous Riemannian metric on $\mathcal{E}$.

At $p = f^n p$, the map $\mathcal{A}_p^n$ has eigenvalues of modulus 1 and is diagonalizable if the 2×2 rotation block has complex eigenvalues, that is the rotation angle $\alpha(p) + \cdots + \alpha(f^{n-1}p)$ does not equal πk. This can be ensured for all periodic points by slightly modifying the function α, if necessary. $\square$

The next result shows that the weaker assumption of existence of invariant conformal structures at periodic points suffices for two-dimensional bundles.

Theorem 6.8. [KS10] *Let $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ be a Hölder continuous linear cocycle over a hyperbolic system (X, f). Suppose that the fibers of $\mathcal{E}$ are two-dimensional.*

(i) *If for each periodic point $p \in X$, the return map $\mathcal{A}_p^n : \mathcal{E}_p \to \mathcal{E}_p$ is diagonalizable over $\mathbb{C}$ and its eigenvalues are equal in modulus, then $\mathcal{A}$ is conformal with respect to a Hölder continuous Riemannian metric on $\mathcal{E}$.*

(ii) *If for each periodic point $p \in X$, the return map $\mathcal{A}_p^n : \mathcal{E}_p \to \mathcal{E}_p$ is diagonalizable over $\mathbb{C}$ and its eigenvalues are of modulus 1, then $\mathcal{A}$ is isometric with respect to a Hölder continuous Riemannian metric on $\mathcal{E}$.*

Outline of the proof. The proof relies on the fact that $\mathcal{A}$ as in the theorem is either conformal or has a one-dimensional continuous invariant sub-bundle, see Corollary 7.5 in the next section. In the first case we immediately obtain (i).

In the latter case, for any point $p = f^n p$ there is an invariant line for $\mathcal{A}_p^n : \mathcal{E}_p \to \mathcal{E}_p$. This implies that the eigenvalues of $\mathcal{A}_p^n$ are real and by the assumption of the theorem, they are either λ, λ or $\lambda, -\lambda$. If $\mathcal{A}$ is orientation preserving, the former is always the case. It follows that $\mathcal{A}_p^n = \lambda \cdot \mathrm{Id}$ since it is diagonalizable. Hence $Q_{\mathcal{A}}(p, n) = 1$ and conformality of $\mathcal{A}$ follows from Theorem 6.5. If $\mathcal{E}$ is not orientable we can pass to a double cover. If $\mathcal{A}$ is orientation reversing, we can consider cocycle $\mathcal{A}^2$ over f^2. Thus we can always obtain an orientation preserving cocycle $\mathcal{A}'$ which is conformal. This implies uniform quasiconformality of the original cocycle $\mathcal{A}$, and conformality of $\mathcal{A}$ follows from Theorem 6.1.

The second part can be establishes as in the proof Theorem 6.5. Indeed, the assumption implies that for any periodic point p the map $\mathcal{A}_p^n$ is conjugate to an orthogonal matrix and hence there exists a constant $C(p)$ such that $\max\{\|\mathcal{A}_p^n\|, \|(\mathcal{A}_p^n)^{-1}\|\} \leq C(p)$ for any period n of p. $\qquad\square$

6.4. **Infinite-dimensional setting.** In [KS18-2] we obtained an analog of Theorem 6.5 for infinite-dimensional setting. The finite dimensional boundedness result was extended in two directions: boundedness and pre-compactness, as the latter does not follow automatically. For a Banach cocycle $\mathcal{A}$, we consider the periodic data set $\mathcal{A}_P$ and the set of all values $\mathcal{A}_X$,

$$\mathcal{A}_P = \{\mathcal{A}_p^k : p = f^k p, \ p \in X, \ k \in \mathbb{N}\} \quad \text{and} \quad \mathcal{A}_X = \{\mathcal{A}_x^n : x \in X, \ n \in \mathbb{Z}\}.$$

Theorem 6.9. [KS18-2] *Let $\mathcal{A}$ be a Hölder continuous Banach cocycle over (X, f).*
  (i) *If there exists a constant C_{per} such that $Q_{\mathcal{A}}(p, k) \leq C_{per}$ whenever $f^k p = p$, then $\mathcal{A}$ is uniformly quasiconformal.*
  (ii) *If the set $\mathcal{A}_P$ is bounded in $(GL(V), d)$, then so is the set $\mathcal{A}_X$.*
  (iii) *If the set $\mathcal{A}_P$ has compact closure in $(GL(V), d)$, then so does the set $\mathcal{A}_X$, moreover, there exists a Hölder continuous family of norms $\|.\|_x$ on V such that $\mathcal{A}_x : (V, \|.\|_x) \to (V, \|.\|_{fx})$ is an isometry for each $x \in X$.*

We note that the closures in (iii) are not the same in general. For example, if $\mathcal{A}$ is a coboundary, i.e., is generated by $A(x) = C(fx) \circ C(x)^{-1}$ for a function $C : X \to GL(V)$, then $\mathcal{A}_P = \{\mathrm{Id}\}$ while $\mathcal{A}_X$ is usually not.

The proofs of uniform quasiconformality and boundedness of the cocycle are similar to the proof of uniform quasiconformality in the first part of of Theorem 6.5. Part (iii) in the infinite-dimensional case requires substantially different arguments. Indeed, Theorem 6.1 and Proposition 6.2 rely on the fact that the space of conformal structures and the space of Euclidean norms have a structure of a symmetric space of nonpositive curvature, but in infinite dimensional case there is no analogous metric structure. Instead, we consider a natural distance on the set of norms but the resulting space is not separable so we work with a small subset. The following general result yields a measurable invariant family of norms and then we show its continuity.

Proposition 6.10. *Let f be a homeomorphism of a metric space X and let $\mathcal{A}$ be a continuous Banach cocycle over (X, f). If the set $\mathcal{A}_X$ has compact closure in $GL(V)$,*

then there exists a bounded Borel measurable family of norms $\|.\|_x$ on V such that $\mathcal{A}_x : (V, \|.\|_x) \to (V, \|.\|_{fx})$ is an isometry for each $x \in X$.

7. Cocycles with one Lyapunov exponent

Now we return to the finite-dimensional case and consider linear cocycles with one Lyapunov exponent, that is, satisfying $\lambda(\mathcal{A}, \mu) = \chi(\mathcal{A}, \mu)$, where $\lambda = \lambda_\ell$ and $\chi = \lambda_1$ are the largest and the smallest Lyapunov exponents given by the Oseledets Multiplicative Ergodic Theorem. Clearly, this is a broader class than conformal or uniformly quasi-conformal cocycles. We considered cocycles with one exponent in [KS09, KS10], and the results in these papers indicated that such cocycles should exhibit some rigidity. In [KS13] we showed that it is true in a strong sense by obtaining a continuous version of Zimmer's Amenable Reduction Theorem.

When $\mathcal{A}$ has more than one Lyapunov exponent, the invariant sub-bundles $\mathcal{E}^i$ given the Oseledets MET are measurable but not necessarily continuous. The next theorem establishes continuity of measurable invariant sub-bundles for fiber bunched cocycles with one exponent. It is a corollary of results by Avila and Viana in [AV10].

Theorem 7.1. *Let (X, f) be a hyperbolic system, let $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ be a β-Hölder fiber bunched linear cocycle over (X, f), and let μ be an ergodic f-invariant probability measure with full support and local product structure. If $\lambda(\mathcal{A}, \mu) = \chi(\mathcal{A}, \mu)$, then any μ-measurable $\mathcal{A}$-invariant sub-bundle of $\mathcal{E}$ is β-Hölder.*

Now we obtain a structure theorem for cocycles with one exponent.

Theorem 7.2 (Continuous Amenable Reduction). [KS13] *Let (X, f) be a hyperbolic system and let $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ be a β-Hölder linear cocycle over f. Suppose that for every periodic point $p = f^n p$ the invariant measure μ_p on its orbit satisfies $\lambda(\mathcal{A}, \mu_p) = \chi(\mathcal{A}, \mu_p)$, that is, all eigenvalues of $\mathcal{A}_p^n$ are equal in modulus. Then there exist a flag of β-Hölder $\mathcal{A}$-invariant sub-bundles*

$$(7.1) \qquad \{0\} = \mathcal{E}^0 \subset \mathcal{E}^1 \subset \cdots \subset \mathcal{E}^{j-1} \subset \mathcal{E}^j = \mathcal{E}$$

and β-Hölder Riemannian metrics on the factor bundles $\mathcal{E}^i/\mathcal{E}^{i-1}$, $i = 1, \ldots, j$, so that the factor-cocycles induced by $\mathcal{A}$ on $\mathcal{E}^i/\mathcal{E}^{i-1}$ are conformal. Moreover, there exists a positive β-Hölder function $\phi : X \to \mathbb{R}$ such that the factor-cocycles of $\phi\mathcal{A}$ on $\mathcal{E}^i/\mathcal{E}^{i-1}$ are isometries.

In the case when $\mathcal{E}_1 = \mathcal{E}$, the cocycle $\mathcal{A}$ itself is conformal on $\mathcal{E}$ with respect to some β-Hölder continuous Riemannian metric. If there are $d = \dim \mathcal{E}_x$ continuous vector fields which give bases for all $\mathcal{E}^i$, then the theorem implies that $\mathcal{A}$ is continuously cohomologous to a cocycle with values in a "standard" maximal amenable subgroup of $GL(d, \mathbb{R})$. However, triviality of $\mathcal{E}$ alone is insufficient for such reduction even if $\mathcal{E} = \mathbb{T}^2 \times \mathbb{R}^2$ since invariant sub-bundles may be non-orientable [S13].

Outline of the proof of Theorem 7.2. The periodic assumptions imply that the cocycle is fiber bunched by Corollary 5.10(ii) and has one exponent for each ergodic measure by

Theorem 5.2. We take μ to be the measure of maximal entropy, or the invariant volume if it exists, and trivialize the bundle $\mathcal{E}$ on a set of full measure, that is, we measurably identify $\mathcal{E}$ with $X \times \mathbb{R}^d$ and view $\mathcal{A}$ as a $GL(d, \mathbb{R})$-valued cocycle. Thus we can use Zimmer's Amenable Reduction [BaPe, Theorem 3.5.9].

Theorem 7.3 (Zimmer's Amenable Reduction). *Let f be an ergodic transformation of a measure space (X, μ) and let $A : X \to GL(d, \mathbb{R})$ be a measurable function.*
Then there exists a measurable function $C : X \to GL(d, \mathbb{R})$ such that the function $B(x) = C^{-1}(fx) A(x) C(x)$ takes values in an amenable subgroup G of $GL(d, \mathbb{R})$.

There are 2^{d-1} *standard* maximal amenable subgroups of $GL(d, \mathbb{R})$ [Mo79]. They correspond to the distinct compositions of d, $d_1 + \cdots + d_k = d$, and each group consists of all block-triangular matrices of the form

$$\begin{bmatrix} B_1 & * & \ldots & * \\ 0 & B_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \ldots & 0 & B_k \end{bmatrix}$$

where each diagonal block B_i is a $d_i \times d_i$ conformal matrix, i.e., a scalar multiple of an orthogonal matrix. Any amenable subgroup of $GL(d, \mathbb{R})$ has a finite index subgroup which is contained in a conjugate of one of these standard subgroups. Thus we may assume that G has a finite index subgroup G_0 which is contained in one of the standard subgroups.

We concentrate on the simplest case when $G_0 = G$. Then the sub-bundle V^i spanned by the first $d_1 + \cdots + d_i$ coordinate vectors in $\mathbb{R}^d$ is B-invariant for $i = 1, \ldots, k$. Denoting $\mathcal{E}_x^i = C(x)V^i$ we obtain the corresponding flag of measurable $\mathcal{A}$-invariant sub-bundles

$$\mathcal{E}^1 \subset \mathcal{E}^2 \subset \cdots \subset \mathcal{E}^k = \mathcal{E} \quad \text{with} \quad \dim \mathcal{E}^i = d_1 + \cdots + d_i.$$

By Theorem 7.1 we may assume that the sub-bundles $\mathcal{E}^i$ are Hölder continuous. Since $B_1(x)$ is a conformal matrix for μ-a.e. x, the push forward by C of the standard conformal structure on V^1 is invariant under the restriction of $\mathcal{A}$ to $\mathcal{E}^1$ and hence β-Hölder continuous by Theorem 6.3.

Similarly, we consider the factor-bundles $\mathcal{E}^i / \mathcal{E}^{i-1}$ over X with the natural induced cocycle $\mathcal{A}^{(i)}$. Since the matrix of the map induced by B on $V^i / V^{i-1} = \mathbb{R}^{d_i}$ is B_i, it preserves the standard conformal structure on $\mathbb{R}^{d_i}$. Pushing it forward by C we obtain a measurable conformal structure τ_i on $\mathcal{E}^i / \mathcal{E}^{i-1}$ invariant under $\mathcal{A}^{(i)}$. The holonomies $H^{\mathcal{A},s}$ and $H^{\mathcal{A},u}$ induce holonomies for $\mathcal{A}^{(i)}$ on $\mathcal{E}^i / \mathcal{E}^{i-1}$. By Lemma 6.4, we conclude that τ_i is essentially invariant under these holonomies and hence is also β-Hölder continuous on X.

Now we outline the argument for the case when $G_0 \neq G$. An example illustrating this case is when G_0 is the full diagonal subgroup and G is its normalizer, which is the finite extension of G_0 that contains all permutations of the coordinate axes.

In the general case, G_0 still has the invariant flag $V^1 \subset V^2 \subset \cdots \subset V^k = \mathbb{R}^d$ with conformal structures on the factors. The elements of G may not preserve this flag, instead there may be several images of it under G:

$$V^1_{(1)} \subset V^2_{(1)} \subset \cdots \subset V^k_{(1)}$$
$$\cdots$$
$$V^1_{(\ell)} \subset V^2_{(\ell)} \subset \cdots \subset V^k_{(\ell)}.$$

These flags are mapped by C to the corresponding measurable flags on $\mathcal{E}$ as before. The sub-bundles in the flags are not $\mathcal{A}$-invariant individually, but $\mathcal{A}$ preserves the union of subspaces at each level of the flag $C(V^i_{(1)}) \cup \cdots \cup C(V^i_{(\ell)})$, $i = 1, \ldots, k-1$. For each level of the flag we show that the union is essentially holonomy invariant and hence Hölder continuous. The union may not split into individual invariant sub-bundles since it may "twist" along non-trivial loops in X. Thus to obtain actual continuous sub-bundles one may need to pass to a special finite cover of X. These sub-bundles are not necessarily invariant as $\mathcal{A}$ may permute them, but they are invariant under an iterate of $\mathcal{A}$. In this way we can obtain the general result Theorem 7.4 below.

Theorem 7.4. [KS13] *Let (X, f) be a hyperbolic system, let μ be an ergodic f-invariant probability measure with full support and local product structure, and let $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ be a β-Hölder continuous linear cocycle over f. Suppose that $\mathcal{A}$ is fiber bunched and $\lambda(\mathcal{A}, \mu) = \chi(\mathcal{A}, \mu)$. Then there exists a finite cover $\tilde{\mathcal{A}} : \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$ of $\mathcal{A}$ and $N \in \mathbb{N}$ such that $\tilde{\mathcal{A}}^N$ satisfies the following property. There exist a flag of β-Hölder continuous $\tilde{\mathcal{A}}^N$-invariant sub-bundles*

$$\{0\} = \tilde{\mathcal{E}}^0 \subset \tilde{\mathcal{E}}^1 \subset \cdots \subset \tilde{\mathcal{E}}^{k-1} \subset \tilde{\mathcal{E}}^k = \tilde{\mathcal{E}}$$

and β-Hölder continuous conformal structures on the factor bundles $\tilde{\mathcal{E}}^i / \tilde{\mathcal{E}}^{i-1}$, $i = 1, \ldots, k$, invariant under the factor-cocycles induced by $\tilde{\mathcal{A}}^N$.

To complete the proof of Theorem 7.2, we claim that the assumption on the periodic data implies that $\mathcal{A}$ is conformal on the sub-bundle spanned by the union $C(V^1_{(1)}) \cup \cdots \cup C(V^1_{(\ell)})$, and then similarly for the induced cocycles on the factors. Indeed, the scalar cocycles that give expansion/contraction of $\mathcal{A}$, more precisely, of an iterate of the lift of $\mathcal{A}$ as above, are all continuously cohomologous by Livšic theorem, since the exponents at each periodic point are the same. This yields uniform quasiconformality and hence conformality of $\mathcal{A}$ on the span. The same reasoning shows that the scalar cocycles of expansion/contraction for the different conformal factors of $\mathcal{A}$ are all cohomologous to one scalar function, whose inverse gives the function ϕ in the theorem. $\square$

We note that the assumption of Theorems 7.2 implies that $\mathcal{A}$ has one exponent with respect to *any* measure, while in Theorem 7.4 the cocycle has one exponent with respect to just *one* measure μ.

Now we apply the theorems to cocycles on bundles with two-dimensional fibers. Then Theorem 7.2 yields the following.

Corollary 7.5. *Let $\mathcal{A}$ and (X, f) be as in Theorem 7.2, and suppose that the fibers of $\mathcal{E}$ are two-dimensional. Then either $\mathcal{A}$ is conformal with respect to a β-Hölder Riemannian metric on $\mathcal{E}$ or $\mathcal{A}$ has a one-dimensional Hölder continuous invariant sub-bundle.*

The proof of Theorem 7.4 can be used to obtain the following two-dimensional result.

Corollary 7.6. [GKS20] *Let $\mathcal{A}$ and (X, f) be as in Theorem 7.4, and suppose that the fibers of $\mathcal{E}$ are two-dimensional. Then at least one of the following holds:*

(1) *$\mathcal{A}$ is conformal with respect to a Hölder continuous Riemannian metric on $\mathcal{E}$;*
(2) *$\mathcal{A}$ preserves a Hölder continuous one dimensional sub-bundle;*
(3) *$\mathcal{A}$ preserves a Hölder continuous field of two transverse lines.*

We note that in the previous corollary every measure has one exponent, which in case (3) would yield conformality.

Using Theorem 7.1 we also obtain the following estimates.

Corollary 7.7 (Polynomial growth of the quasiconformal distortion and norm)**.**
Let $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ be a Hölder continuous linear cocycle over a hyperbolic system (X, f). Suppose that for every f-periodic point p the invariant measure μ_p on its orbit satisfies $\lambda(\mathcal{A}, \mu_p) = \chi(\mathcal{A}, \mu_p)$. Then there exists $m < \dim \mathcal{E}_x$ and C such that

$$Q_{\mathcal{A}}(x, n) \leq Cn^{2m} \quad \text{for all } x \in X \text{ and } n \in \mathbb{Z}.$$

Moreover, if $\lambda(\mathcal{A}, \mu_p) = \chi(\mathcal{A}, \mu_p) = 0$ for every μ_p, then there exists $m < \dim \mathcal{E}_x$ and C such that

$$\|\mathcal{A}_x^n\| \leq C|n|^m \quad \text{for all } x \in X \text{ and } n \in \mathbb{Z}.$$

One can take $m = j - 1$, which is the number of non-trivial sub-bundles in (7.1).

7.1. Classification of $GL(2, \mathbb{R})$-valued cocycles with one exponent.

The structure Theorems 7.2 and 7.4 and Corollary 7.5 can be used to study cohomology of cocycles. First such results were obtained in the two dimensional case before the theorems in higher dimensions were established. In [KS10] we proved criteria for conformality and isometry, and hence cohomology to a conformal or orthogonal cocycle, see Theorem 6.8. Then a complete classification up to Hölder cohomology of $GL(2, \mathbb{R})$-valued cocycles with one exponent at each periodic point was obtained in [S13]. It showed that such cocycles can be viewed as either elliptic or parabolic.

Theorem 7.8. [S13] *Let $f : X \to X$ is a transitive Anosov diffeomorphism, and let $\mathcal{A}$ be an orientation-preserving $GL(2, \mathbb{R})$-valued cocycle over (X, f). Suppose that for each periodic point $p = f^n p$ in X, the eigenvalues of the matrix A_p^n are equal in modulus. Then $\mathcal{A}$ belongs to exactly one of the five types below.*

I. *If $\mathcal{A}$ preserves exactly one Hölder continuous sub-bundle, which is orientable, then $\mathcal{A}$ is Hölder cohomologous to a cocycle $\mathcal{A}'$ with the generator*
$A'(x) = k(x) \begin{bmatrix} 1 & \alpha(x) \\ 0 & 1 \end{bmatrix}$, *where $k(x) \neq 0$ and α is not cohomologous to 0.*

I′. *If $\mathcal{A}$ preserves exactly one Hölder continuous sub-bundle, which is not orientable, then there exists a cocycle $\mathcal{A}'$ as in I such that the lifts of $\mathcal{A}$ and $\mathcal{A}'$ to a double cover are Hölder cohomologous.*

II. *If $\mathcal{A}$ preserves more than one orientable Hölder continuous sub-bundle, then $\mathcal{A}$ is Hölder cohomologous to $\mathcal{A}'$ with $A'(x) = k(x) \cdot Id$, where $k(x) \neq 0$.*

II′. *If $\mathcal{A}$ preserves more than one non-orientable Hölder continuous sub-bundle, then there exists a cocycle $\mathcal{A}'$ as in II such that the lifts of $\mathcal{A}$ and $\mathcal{A}'$ to a double cover are Hölder cohomologous.*

III. *If $\mathcal{A}$ has no invariant Hölder continuous sub-bundles, then $\mathcal{A}$ is Hölder cohomologous to $\mathcal{A}'$ with $A'(x) = k(x)\,R(\alpha(x))$, where $k(x) > 0$, $R(\alpha(x))$ is the rotation by $\alpha(x)$, and $\alpha : X \to \mathbb{R}/2\pi\mathbb{Z}$ is such that α is not cohomologous to 0 in $\mathbb{R}/\pi\mathbb{Z}$.*

Examples of cocycles as in I′ and I″ were also constructed in [S13]. We note that for cocycles as in the theorem, any measurable invariant sub-bundle is Hölder continuous, thus measurable sub-bundles can be considered in the statement.

Explicit necessary and sufficient conditions for cohomology of "model" cocycles within each class were also obtained in [S13], cf. Proposition 4.7. In particular these results yielded the following.

Corollary 7.9. [S13] *Let μ be an f-invariant ergodic measure with full support and local product structure, and let $\mathcal{A}$ and $\mathcal{B}$ be two cocycles as in Theorem 7.8. Then any μ-measurable conjugacy between $\mathcal{A}$ and $\mathcal{B}$ is Hölder continuous. In particular, the Hölder classification in the theorem coincides with the measurable one.*

We note that cocycles of types I and I′ are not uniformly quasiconformal and hence Theorem 4.14 on continuity of measurable conjugacy does not apply to them.

8. Cocycles over partially hyperbolic diffeomorphisms

Many of the ideas and techniques that were used for hyperbolic systems extend to partially hyperbolic setting.

A diffeomorphism f of X is called *partially hyperbolic* if there exist a nontrivial Df-invariant splitting of the tangent bundle $TX = E^s \oplus E^c \oplus E^u$, a Riemannian metric on X, and continuous functions $\nu < 1 < \hat{\nu}$, γ, $\hat{\gamma}$ such that for any $x \in X$ and any unit vectors $v^{s/c/u}$ in $E^{s/c/u}(x)$,

(8.1) $$\|Df(v^s)\| < \nu(x) < \gamma(x) < \|Df(v^c)\| < \hat{\gamma}(x) < \hat{\nu}(x) < \|Df(v^u)\|.$$

The sub-bundles E^s, E^u, and E^c are called stable, unstable, and center, and E^s and E^u are tangent to the foliations W^s and W^u. A partially hyperbolic diffeomorphism is called *accessible* if any two points in X can be connected by an *su*-path, i.e., a concatenation of finitely many subpaths lying in a single leaf of W^s or W^u, and f is called *center bunched* if $\nu < \gamma\hat{\gamma}^{-1}$ and $\hat{\nu}^{-1} < \gamma\hat{\gamma}^{-1}$.

Assumptions. In this section, f is an accessible center bunched partially hyperbolic C^2 diffeomorphism preserving a volume μ, and $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ is a β-Hölder linear cocycle over (X, f). The volume μ is ergodic for such f by [BuW10].

Problems of continuity for invariant objects and measurable conjugacies are natural in this setting. The following theorem extends the corresponding results for hyperbolic systems.

Theorem 8.1. [KS13, ASV13] *Let f, μ and $\mathcal{A}$ be as in the Assumptions.*

   (i) *If $\mathcal{A}$ is fiber bunched, then any $\mathcal{A}$-invariant μ-measurable conformal structure on $\mathcal{E}$ is continuous, that is, coincides μ-a.e. with a continuous one.*

  (ii) *If $\mathcal{A}$ is uniformly quasiconformal then it preserves a continuous conformal structure on $\mathcal{E}$, equivalently, $\mathcal{A}$ is conformal with respect to a continuous Riemannian metric on $\mathcal{E}$.*

 (iii) *If $\mathcal{A}$ is fiber bunched and has one Lyapunov exponent with respect to μ, then any μ-measurable $\mathcal{A}$-invariant sub-bundle of $\mathcal{E}$ is continuous.*

Note that only continuity of measurable objects is obtained in the partially hyperbolic case. Hölder continuity can be proved under a stronger accessibility assumption.

Outline of the proof. Fibers bunching for cocycles over partially hyperbolic systems is defined in the same way as for hyperbolic ones, see Definition 3.2. It implies existence of stable and unstable holonomies for $\mathcal{A}$ as in Proposition 3.3. Lemma 6.4 also extends directly and yields essential invariance of a measurable conformal structure τ under the stable and unstable holonomies of $\mathcal{A}$. In the proof of Theorem 8.1, the local product structure of stable and unstable manifolds is replaced by accessibility. In this case the global continuity of τ on X follows from [ASV13, Theorem E] or [W13, Theorem 4.2]. Part (ii) follows from part (i) and Proposition 6.2. Essential invariance of the sub-bundle in part (iii) is obtained using results in [AV10, ASV13]. $\square$

Using Theorem 8.1 we extended Theorem 7.4 on structure of cocycles with one exponent to partially hyperbolic setting [KS13].

Theorem 8.2. [KS13] *Let f, μ and $\mathcal{A}$ be as in the Assumptions. Suppose that $\mathcal{A}$ is fiber bunched and $\lambda(\mathcal{A}, \mu) = \chi(\mathcal{A}, \mu)$ for the invariant volume μ.*

Then there exists a finite cover $\tilde{\mathcal{A}} : \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$ of $\mathcal{A}$ and $N \in \mathbb{N}$ such that $\tilde{\mathcal{A}}^N$ satisfies the following property. There exist a flag of continuous $\tilde{\mathcal{A}}^N$-invariant sub-bundles

$$(8.2) \qquad \{0\} = \tilde{\mathcal{E}}^0 \subset \tilde{\mathcal{E}}^1 \subset \cdots \subset \tilde{\mathcal{E}}^{k-1} \subset \tilde{\mathcal{E}}^k = \tilde{\mathcal{E}}$$

and continuous conformal structures on the factor bundles $\tilde{\mathcal{E}}^i / \tilde{\mathcal{E}}^{i-1}$, $i = 1, \ldots, k$, invariant under the factor-cocycles induced by $\tilde{\mathcal{A}}^N$.

Now we consider cohomology of linear cocycles over partially hyperbolic systems. Continuity of a measurable conjugacy for real-valued cocycles was obtained by Wilkinson in [W13]. The next theorem extends this result to linear cocycles.

Theorem 8.3. [KS16] *Let f and μ be as in the Assumptions, and let $\mathcal{A}$ and $\mathcal{B}$ be Hölder continuous linear cocycles over (X, f). If $\mathcal{A}$ is fiber bunched and $\mathcal{B}$ is uniformly quasiconformal, then any μ-measurable conjugacy C between $\mathcal{A}$ and $\mathcal{B}$ coincides on a set of full volume with a continuous conjugacy.*

The proof uses the techniques of the hyperbolic setting as in the proof of Theorem 4.14 to obtain essential Hölder continuity along stable and unstable foliations and then concludes global continuity using results from [ASV13, W13]. Global Hölder continuity for scalar cocycles was obtained in [W13]. For linear cocycles this is only known under a stronger accessibility assumption [KS16].

Partially hyperbolic systems may have no periodic points. In analogs of Questions 4.4 and 4.5 for these systems, the role of periodic data is played by *cycle weights (or functionals)*. A cycle weight is a composition of holonomies of a cocycle along a closed su-path. More precisely, let $H^{\mathcal{A},s}$ and $H^{\mathcal{A},u}$ be the stable and unstable holonomies for a cocycle $\mathcal{A}$. Let $P = P_{x_0} = \{x_0, x_1, \ldots, x_{k-1}, x_k = x_0\}$ be an su-cycle, that is a closed su-path in X. The *cycle weight* of P is

$$\mathcal{H}_{x_0}^{\mathcal{A},P} = H_{x_{k-1},x_0} \circ \cdots \circ H_{x_1,x_2} \circ H_{x_0,x_1},$$

where $H_{x_i,x_{i+1}} = H_{x_i,x_{i+1}}^{s/u}$ if $x_{i+1} \in W^{s/u}(x_i)$.

In case of real-valued cocycles, $\mathcal{H}_{x_0}^{\mathcal{A},P}$ is also called a cycle functional. This notion was introduced in [KtKo96] to show that for systems with strong accessibility, a real-valued cocycle has zero cycle weights if and only if it is cohomologous to a constant cocycle. This was extended in some cases to non-commutative cocycles [KtNT00] but, as we showed in [KS16], having trivial cycle weights is not necessary in general for continuous cohomology to a constant cocycle. In [KS16] we obtained necessary and sufficient conditions for cohomology of a cocycle to a constant one, as well as the following result for cohomology of two arbitrary cocycles in terms of cycle weights.

Theorem 8.4. *Let f be an accessible center bunched partially hyperbolic diffeomorphism. Let $\mathcal{A}$ and $\mathcal{B}$ be β-Hölder linear or Banach cocycles over (X, f) with holonomies $H^{\mathcal{A}}$ and $H^{\mathcal{B}}$. Suppose that there exist a fixed point x_0 and $C_{x_0} \in GL(V)$ such that*

$$\mathcal{A}_{x_0} = C_{x_0} \circ \mathcal{B}_{x_0} \circ C_{x_0}^{-1} \quad and \quad \mathcal{H}_{x_0}^{\mathcal{A},P} = C_{x_0} \circ \mathcal{H}_{x_0}^{\mathcal{B},P} \circ C_{x_0}^{-1} \text{ for every su-cycle } P_{x_0}.$$

Then there exists a continuous conjugacy C between $\mathcal{A}$ and $\mathcal{B}$ with $C(x_0) = C_{x_0}$.

For the case when x_0 is not fixed, we established similar conditions that guarantee existence of a a continuous conjugacy C with $C(x_0) = C_{x_0}$.

9. Applications to rigidity problems for hyperbolic systems

9.1. **Local rigidity.** It is well-known that if g is an Anosov diffeomorphism and f is sufficiently C^1 close to g, then f is also Anosov and f is topologically conjugate to g, that is, there is a homeomorphism h of X such that $g = h^{-1} \circ f \circ h$. In general, h is only

Hölder continuous. A necessary condition for it to be C^1 is that the derivatives of the return maps of f and g at the corresponding periodic points are conjugate, specifically,

$$D_p g^n = (D_p h)^{-1} \circ D_{h(p)} f^n \circ D_p h \quad \text{whenever } p = f^n p.$$

An Anosov diffeomorphism g is said to be *locally rigid* if for any C^1-small perturbation f this condition is also sufficient for the conjugacy to be a C^1 diffeomorphism. The problem of local rigidity has been extensively studied and Anosov diffeomorphisms with one-dimensional stable and unstable sub-bundles were shown to be locally rigid [L87, LM88, L92]. In higher dimensions, examples where the periodic condition is not sufficient were constructed by de la Llave [L92, L02]. We recall that L is *irreducible* if it has no rational invariant subspaces, equivalently, if its characteristic polynomial is irreducible over $\mathbb{Q}$. Examples in [G08] showed that irreducibility of L is a necessary condition for local rigidity, except when L is conformal on the stable and unstable sub-bundles. In the latter case local rigidity was obtained for some systems in [L02] and for broader classes [KS03, KS09] as a corollary of global rigidity results, see Theorem 9.7 and Remark 9.8.

Local rigidity was proved in [G08] for an irreducible Anosov toral automorphism $L : \mathbb{T}^d \to \mathbb{T}^d$ with real eigenvalues of distinct moduli, as well as for some nonlinear systems with similar structure. In [GKS11, S15] we used results on conformality and cohomology of linear cocycles to establish the following local rigidity results.

Theorem 9.1. [S15, GKS11] *Let $L : \mathbb{T}^d \to \mathbb{T}^d$ be an irreducible Anosov automorphism and let f be a C^1-small perturbation of L such that for each periodic point $p = f^n p$ there is a matrix $C(p)$ such that $D_p f^n = C(p) \circ L^n \circ C(p)^{-1}$. Suppose that either*

   (i) *No three of the eigenvalues of L have the same modulus* [GKS11], *or*

   (ii) *$C(p)$ is Hölder continuous at a periodic point p_0, that is,*
      $d(C(p), C(p_0)) \le c\, dist(p, p_0)^\beta$ for every periodic point p [S15].

Then f is $C^{1+\text{Hölder}}$ conjugate to L.

We note that f in this theorem has an invariant volume. Indeed, the conjugacy assumption implies $\det D_p f^n = \det L^n = 1$ whenever $p = f^n p$ and existence of density which gives an invariant volume follows from Livšic Theorem 4.1(i).

Outline of the proof of Theorem 9.1. The proofs of (i) and (ii) differ only in the way we obtain conformality of Df on certain invariant sub-bundles. Conformality of these restrictions plays an important role in establishing smoothness of the conjugacy.

We denote by $E^{u,L}$ the unstable sub-bundle of L. Let $1 < \rho_1 < \cdots < \rho_\ell$ be the distinct moduli of the unstable eigenvalues of L, and let $E^{u,L} = E_1^L \oplus \cdots \oplus E_\ell^L$ be the corresponding splitting of the unstable sub-bundle. Since f is C^1 close to L, f is also Anosov, and its unstable sub-bundle $E^{u,f}$ splits into a direct sum of ℓ invariant Hölder continuous sub-bundles close to the corresponding sub-bundles for L: $E^{u,f} = E_1^f \oplus \cdots \oplus E_\ell^f$. Let $\mathcal{A}_i = L|E_i^L$ and $\mathcal{B}_i = Df|E_i^f$.

Since L is irreducible, all its eigenvalues are simple and thus L and its the restrictions $\mathcal{A}_i$ are diagonalizable over $\mathbb{C}$. Since the eigenvalues of $\mathcal{A}_i$ are of the same modulus, $\mathcal{A}_i$ is

conformal in some norm. Since $C(p)$ maps $E_i^L(p)$ to $E_i^f(p)$, the cocycle $\mathcal{B}_i$ has conjugate periodic data, and hence is conformal at the periodic points.

In [GKS11], conformality of $\mathcal{B}_i$ at the periodic points together with the assumption that the sub-bundles E_i^L and E_i^f are either one- or two-dimensional allows us to conclude that, by Theorem 6.8, the cocycle $\mathcal{B}_i$ is conformal. In higher dimensions, conformality at the periodic points does not imply conformality. In (ii) we can use Corollary 4.12 to conclude that $\mathcal{A}_i$ and $\mathcal{B}_i$ are Hölder continuously cohomologous, and so in particular $\mathcal{B}_i$ is also conformal.

After conformality of Df on each sub-bundle E_i^f is obtained, we consider the topological conjugacy h close to the identity between L and f. If h maps foliation W_i^f tangent to E_i^f to the corresponding linear foliation W_i^L, the conformality allows us to obtain that h is Lipschitz along the leaves of W_i^f. This also uses the fact that $\mathcal{A}_i$ and $\mathcal{B}_i$, or at least their norms, are continuously cohomologous. Then we can differentiate the conjugacy equation $h \circ f = L \circ h$ almost everywhere with respect to volume. The derivative $D(h|W_i^f)$ is then a measurable conjugacy between $\mathcal{A}_i$ and $\mathcal{B}_i$ and hence is Hölder by Theorem 4.14. This implies that h is $C^{1+\text{Hölder}}$ along W_i^f, and having this for each foliation W_i^f yields that h is $C^{1+\text{Hölder}}$ along $W^{u,f}$. Then regularity of h along $W^{s,f}$ is obtained similarly and regularity on $\mathbb{T}^d$ follows.

Establishing that $h(W_i^f) = W_i^L$ for each i is done using an inductive process within stable and unstable foliations. The base is given by the slowest foliation W_1. The inductive step also uses Hölder conjugacy between $\mathcal{A}_i$ and $\mathcal{B}_i$ established at the previous steps, as well as their conformality, to obtain conformality of the holonomies of certain foliations between the leaves of W_i^f. $\qquad\square$

Since L is linear, its Lyapunov exponents are the same for all invariant measures. The conjugacy of periodic data of f and L implies that Lyapunov exponents of f with respect to any f-invariant measure μ equal to the Lyapunov exponents of L by Theorem 5.2 on their periodic approximation.

Saghin and Yang obtained smoothness of the conjugacy h for a volume preserving perturbation f of an irreducible L if f and L have the same Lyapunov exponents with respect to the volume and all Lyapunov exponents are *simple* [SaY19]. The following theorem allows for double Lyapunov exponents and extends the Saghin-Yang result to a much broader class of irreducible hyperbolic automorphisms. We note, however, that these results rely on the fact that volume is the measure of maximal entropy for L, and they do not hold in general when L is replaced by a non-linear system.

Theorem 9.2. [GKS20] *Let $L : \mathbb{T}^d \to \mathbb{T}^d$ be an Anosov automorphism such that no three of its eigenvalues have the same modulus and L^4 is irreducible. Let f be a volume-preserving C^2 diffeomorphism of $\mathbb{T}^d$ sufficiently C^1-close to L. If the Lyapunov exponents of f with respect to the volume are the same as the Lyapunov exponents of L, then f is $C^{1+\text{Hölder}}$ conjugate to L.*

In fact, it suffices to assume that L is irreducible and have no pairs of eigenvalues of the form $\lambda, -\lambda$ or $i\lambda, -i\lambda$, where λ is real, which follows if L^4 is irreducible. This

assumption implies that L has no invariant lines or pairs of lines. The same property holds for the derivative at a fixed point for any small perturbation f. This implies that for the cocycle $\mathcal{B}_i = D(h|W_i^f)$, as in the proof of Theorem 9.1, options (2) and (3) in Corollary 7.6 are impossible and hence $\mathcal{B}_i$ is conformal. The other significant difference from the proof of Theorem 9.1 is that cohomology of the norms of $\mathcal{A}_i$ and $\mathcal{B}_i$ is obtained using the Jacobian of the conjugacy h restricted to W_i^f. The latter relies on establishing that h maps the conditional measures of the volume for f on W_i^f to those for L. Otherwise, the structure of the proof is similar.

9.2. Smoothness of the stable/unstable sub-bundles and global rigidity.

The results on conformality of linear cocycles in Section 6 were motivated in part by the study of rigidity for conformal Anosov systems. The initial results were established in [S02] for the restrictions of Df to the stable and unstable sub-bundles. They allowed to obtain regularity of the foliations and *global rigidity,* that is existence of a conjugacy to an algebraic model, for some classes of Anosov systems.

Theorem 9.3. [S02] *Let f be a transitive C^∞ Anosov diffeomorphism of X. Suppose that Df is uniformly quasiconformal on E^s with $\dim E^s \geq 2$. Then*

  (i) *Df is conformal with respect to a Riemannian metric on E^s which is Hölder continuous on X and C^∞ along the leaves of the stable foliation.*

  (ii) *The holonomy maps of the unstable foliation are conformal and E^u is C^∞.*

A similar result holds for a topologically mixing C^∞ Anosov flow that is uniformly quasiconformal on the strong stable sub-bundle E^s. It yields conformality on E^s, conformality of the holonomies of the weak stable foliation, and C^∞ regularity of the weak stable sub-bundle [S02]. The proof is also similar.

Outline of the proof of Theorem 9.3.

(i) Since Df is uniformly quasiconformal on E^s, it preserves a Hölder continuous conformal structure τ on E^s. We show its smoothness along the leaves using a non-stationary linearization. Taking a smooth normalization of τ, we obtain a desired Riemannian metric on E^s.

The statement below is a generalization of a one-dimensional result established by Katok and Lewis in [KtL91].

Proposition 9.4 (Non-stationary linearization). [S02] *Let f be a diffeomorphism of a compact Riemannian manifold X, and let W be a continuous invariant foliation with C^∞ leaves. Suppose that $\|Df|_{TW}\| < 1$, and there exist $C > 0$ and $0 < \theta < 1$ such that*

$$\| (Df^n|_{T_xW})^{-1} \| \cdot \| Df^n|_{T_xW} \|^2 \leq C\theta^n \quad \text{for all } x \in X \text{ and } n \in \mathbb{N}.$$

Then for every $x \in X$ there exists a C^∞ diffeomorphism $h_x : W(x) \to T_xW$ such that $h_{fx} \circ f = Df_x \circ h_x$, $h_x(x) = 0$, $D_xh_x = Id$, and h_x depends continuously on x in C^∞ topology.

Applying the proposition to f yields a non-stationary linearization $\{h_x\}$ along W^s.

For each $x \in X$ we extend the conformal structure $\tau(x)$ to a constant conformal structure σ on TE_x^s, that is, for each $t \in E_x^s$ we denote by $\sigma(t)$ the push forward of $\tau(x)$ by the translation from 0 to t.

Lemma 9.5. *Each $h_x : W^s(x) \to E_x^s$ is conformal with respect to τ and σ, that is, $h_x(\tau(y)) = \sigma(h_x(y))$ for any $y \in W^s(x)$, and hence τ is C^∞ along the leaves of W^s.*

Proof. By continuity of τ and the properties of h_x, for each $\varepsilon > 0$ there exists $n > 0$ such that
$$\text{dist}(h_{f^n x}(\tau(f^n y)),\, \sigma(h_{f^n x}(f^n(y)))) < \varepsilon.$$
To obtain the following equalities, we note that Df^n induces an isometry between the spaces of conformal structures, τ is f-invariant, σ is Df-invariant, and $h_x(y) = Df^{-n}(h_{f^n x}(f^n y))$. Thus,
$$
\begin{aligned}
\varepsilon &> \text{dist}\,(\,h_{f^n x}(\tau(f^n y)),\, \sigma(h_{f^n x}(f^n y))\,) \\
&= \text{dist}\,(\,Df^{-n}(h_{f^n x}(\tau(f^n y))),\, Df^{-n}(\sigma(h_{f^n x}(f^n y)))\,) \\
&= \text{dist}\,(\,Df^{-n}(h_{f^n x}(f^n(\tau(y)))),\, \sigma(Df^{-n}(h_{f^n x}(f^n y)))\,) \\
&= \text{dist}\,(\,h_x(\tau(y)),\, \sigma(h_x(y))\,).
\end{aligned}
$$
It follows that $h_x(\tau(y)) = \sigma(h_x(y))$. Since h is a C^∞ diffeomorphism and σ is constant, τ is C^∞ along the leaves of W^s. $\qquad\square$

(ii) Let x and y be two nearby points in X. We consider the holonomy map of the unstable foliation $\mathcal{H}_{x,y} : W^s_{\text{loc}}(x) \to W^s_{\text{loc}}(y)$:
$$z \in W^s_{\text{loc}}(x) \mapsto \mathcal{H}_{x,y}(z) = W^s_{\text{loc}}(y) \cap W^u_{\text{loc}}(z).$$
First we show that the holonomy maps $\mathcal{H}_{x,y}$ are conformal, i.e. $\mathcal{H}_{x,y}(\tau(z)) = \tau(\mathcal{H}_{x,y}(z))$. By the C^r-section Theorem [HPuSh], conformality of Df on E^s implies that the unstable sub-bundle is C^1, and hence the holonomy maps are uniformly C^1 [PSW97]. It suffices to consider the case when $z = x$ and $y \in W^u(x)$ so that $y = \mathcal{H}_{x,y}(z)$. We iterate by f^{-1} and note that $\mathcal{H}_{x,y} = f^n \circ \mathcal{H}_{f^{-n}x, f^{-n}y} \circ f^{-n}$ and that $D_{f^{-n}x}\mathcal{H}_{f^{-n}x, f^{-n}y}$ is close to identity for large n since $f^{-n}y$ is close to $f^{-n}x$. Since τ is continuous, $\tau(f^{-n}x)$ is close to $\tau(f^{-n}y)$. Thus, $\mathcal{H}_{f^{-n}x, f^{-n}y}(\tau(f^{-n}x))$ is close to $\tau(f^{-n}y)$. Since Df^n induces an isometry between the spaces of conformal structures on $E^s(f^{-n}y)$ and on $E^s(y)$, we conclude that $\mathcal{H}_{x,y}(\tau(x))$ is close to $\tau(y)$. By letting $n \to \infty$, we obtain $\mathcal{H}_{x,y}(\tau(z)) = \tau(\mathcal{H}_{x,y}(z))$.

Now we show that $\mathcal{H}_{x,y}$ are C^∞. Using the non-stationary linearization coordinates we view it as the map
$$G_{x,y} = h_y \circ \mathcal{H}_{x,y} \circ h_x^{-1} : \ E_x^s \to E_y^s,$$
Then $G_{x,y}$ is a conformal C^1 diffeomorphism of $\mathbb{R}^d$ defined on a neighborhood of 0, and hence it is C^∞. Indeed, if $d > 2$, it is Möbius, and if $d = 2$ it is complex analytic. Therefore $\mathcal{H}_{x,y}$ is C^∞ and it follows that E^u is C^∞ [PSW97]. $\qquad\square$

We say that an Anosov diffeomorphism is uniformly quasiconformal if it is uniformly quasiconformal on both E^u and E^s. Theorem 9.3 implies that for such diffeomorphisms

both E^u and E^s are C^∞. In general, smoothness of both E^u and E^s is not known to imply smooth conjugacy to algebraic model, although it is conjectured. However, the implication is known to hold under additional assumptions, for example that f is symplectic [BFL92] or that f preserves a smooth affine connection [BL93]. Thus we obtain the following result for a symplectic diffeomorphism f. For such f, conformality on the stable sub-bundle implies conformality on the unstable one, and vice versa.

Theorem 9.6. [S02] *Let f be a C^∞ symplectic Anosov diffeomorphism of a compact manifold X, $\dim X \geq 4$. Suppose that f is uniformly quasiconformal on the unstable sub-bundle. Then a lift of f to a finite cover of X is C^∞ conjugate to an Anosov automorphism of a torus.*

Without symplecticity, we obtained the following general result in [KS03].

Theorem 9.7 (Global rigidity of uniformly quasiconformal Anosov diffeomorphisms)**.** [KS03] *Let f be a transitive C^∞ Anosov diffeomorphism of a compact manifold X which is uniformly quasiconformal on the stable and unstable sub-bundles. Suppose either that both sub-bundles have dimension at least three, or that they have dimension at least two and X is an infranilmanifold. Then a lift of f to a finite cover of X is C^∞ conjugate to an Anosov automorphism of a torus.*

Outline of the proof. Theorem 9.3 implies that both E^u and E^s are C^∞ and there is a C^∞ Riemannian metric on X for which f is conformal on both E^u and E^s. Under the assumptions of the theorem, we showed that the holonomy maps $\mathcal{H}_{x,y}$ are globally defined affine conformal maps between the leaves. Then we proved that the non-stationary linearization coordinates h_x^s and h_x^u depend C^∞ on x. This allowed us to construct a smooth invariant affine connection. Existence of such a connection together with the smoothness of E^u and E^s implies the conjugacy by the result in [BL93]. $\square$

Remark 9.8. *Uniform quasiconformality or conformality of f can be established from its periodic data. Results of this type were obtained in* [KS03, KS09], *and later extended to general cocycles in Theorems 6.5 and 6.8.*

9.3. **Global rigidity of Anosov flows.** Continuous time conformal Anosov systems were studied by Kanai in the special case of geodesic flows. He proved in [Kn93] that the geodesic flow of a compact Riemannian manifold of negative curvature of dimension at least three is C^2 conjugate to the geodesic flow of a manifold of constant negative curvature under the assumption that either the flow preserves a continuous conformal structure on the strong stable sub-bundle, or the flow satisfies 1/2 pinching and preserves a bounded measurable conformal structure on the strong stable sub-bundle. The next theorem generalizes this result to the case of contact flows.

Theorem 9.9. [S02] *Let φ^t be a C^∞ contact Anosov flow on a compact manifold $\mathcal{M}$, $\dim \mathcal{M} \geq 5$. If the flow is uniformly quasiconformal on the strong stable sub-bundle, then it is essentially, that is up to a time change of a specific form, C^∞ conjugate to the geodesic flow of a manifold of constant negative curvature.*

This theorem follows from the analog of Theorem 9.3 for flows and the result of Benoist, Foulon, and Labourie on rigidity of contact Anosov flows with smooth stable and unstable sub-bundles in [BFL92]. In a particular case when φ^t is a geodesic flow of a compact Riemannian manifold $\mathcal{N}$ of negative curvature of dimension at least three, the theorem yields that φ^t is conjugate to the geodesic flow of a manifold of constant negative curvature. Then it follows from the main theorem of Besson, Courtois, and Gallot in [BCG95] that $\mathcal{N}$ is a manifold of constant negative curvature.

We showed in [S02] that *any* smooth time change of a uniformly quasiconformal Anosov flow is also uniformly quasiconformal. Therefore, if the flow is not assumed contact then it could be any smooth time change of the geodesic flow of a hyperbolic manifold or of the suspension flow of a conformal hyperbolic automorphism of a torus. Rigidity of uniformly quasiconformal Anosov diffeomorphisms and flows was further studied by Fang in [Fa04, Fa07]. In particular, the following classification was obtained in higher dimensions.

Theorem 9.10. [Fa07] *Let φ^t be a C^∞ topologically transitive uniformly quasiconformal Anosov flow such that the strong stable and unstable sub-bundles E^u and E^s are at least three dimensional. Then up to finite covers, φ^t is C^∞ conjugate to a time change of either the geodesic flow of a hyperbolic manifold or the suspension of a hyperbolic automorphism of a torus.*

In the context of geodesic flows, stronger results on rigidity of Lyapunov spectrum were recently obtained by Butler [Bt17]. They rely on techniques from [S02] and analogs of structure results for cocycles with one Lyapunov exponent, Theorems 7.2 and 7.4. We note that 1/4-pinching of the sectional curvature for geodesic flows ensures fiber bunching of stable and unstable differential for the geodesic flow.

Theorem 9.11. [Bt17] *Let $\mathcal{M}$ be a closed negatively curved Riemannian manifold with $\dim \mathcal{M} \geq 3$.*

  (i) *If at each periodic orbit the geodesic flow φ^t has one unstable Lyapunov exponent, then $\mathcal{M}$ has constant curvature.*

  (ii) *Suppose that $\mathcal{M}$ has relatively 1/4-pinched sectional curvatures. Let μ be an ergodic measure with full support and local product structure invariant under the geodesic flow φ^t. If μ has one unstable Lyapunov exponent then $\mathcal{M}$ has constant curvature.*

Further results on Lyapunov spectrum rigidity for geodesic flows on negatively curved symmetric spaces were obtained in [Bt19].

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