FIBER BUNCHING AND COHOMOLOGY FOR BANACH COCYCLES OVER HYPERBOLIC SYSTEMS

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ABSTRACT. We consider Hölder continuous cocycles over hyperbolic dynamical systems with values in the group of invertible bounded linear operators on a Banach space. We show that two fiber bunched cocycles are Hölder continuously cohomologous if and only if they have Hölder conjugate periodic data. The fiber bunching condition means that non-conformality of the cocycle is dominated by the expansion and contraction in the base system. We show that this condition can be established based on the periodic data of a cocycle. We also establish Hölder continuity of a measurable conjugacy between a fiber bunched cocycle and one with values in a set which is compact in strong operator topology.

1. INTRODUCTION AND STATEMENTS OF THE RESULTS

Cocycles play an important role in dynamics. Cohomology of real-valued and, more generally, group-valued cocycles over hyperbolic systems has been extensively studied starting with the seminal work of A. Livšic [Liv71], see [KtN] for an overview. The study has been focused on obtaining cohomology of two cocycles from their periodic data, i.e. the values at the periodic points of the base system, and on regularity of transfer map, or conjugacy, between two cocycles. Livšic resolved the case of cocycles with values in $\mathbb{R}$ or an abelian group and made some progress for more general groups. For smooth dynamical systems, the differential and its restrictions to invariant sub-bundles give important examples of cocycles. This motivated in part the extensive research of $GL(n, \mathbb{R})$ and Lie group valued cocycles. Cohomology problems for cocycles with values in non-abelian groups are much more difficult. The case when one of the cocycles is the identity has been studied most and by now is relatively well understood, see for example [Liv72, NT95, PWa01, LW10, K11, GG]. The cohomology problem for two arbitrary cocycles does not reduce to this special case for non-abelian groups. This problem was first considered in [Pa99] for compact groups and in [Sch99] for cocycles with "bounded distortion". The most general results for $GL(n, \mathbb{R})$ were established in [S15] for fiber bunched cocycles. One of these results was also independently obtained in [B15].

The infinite dimensional case is more difficult and so far is much less developed. For two arbitrary cocycles with values in the group of diffeomorphisms of a compact manifold, higher regularity of the conjugacy was studied in [NT98] and recently cohomology of cocycles with equal periodic data was obtained in [BK16] under a certain bunching assumption on both cocycles. In this paper we extend the results for finite dimensional linear cocycles to the infinite-dimensional setting. We consider cocycles with values in the group of invertible operators on a Banach space V. The simplest examples are given by random and Markov sequences of operators. They correspond to locally constant cocycles over subshifts of finite type.

The space $L(V)$ of bounded linear operators on V is a Banach space equipped with the operator norm $\|A\| = \sup\{\|Av\| : v \in V,\ \|v\| \leq 1\}$. The open set $GL(V)$ of invertible elements in $L(V)$ is a topological group and a complete metric space with respect to the metric

$$(1.1) \qquad\qquad d(A, B) = \|A - B\| + \|A^{-1} - B^{-1}\|.$$

Definition 1.1. *Let f be a homeomorphism of a compact metric space X and let A be a function from X to $(GL(V), d)$. The* Banach cocycle over f generated by A is *the map $\mathcal{A} : X \times \mathbb{Z} \to G$ defined by $\mathcal{A}(x, 0) = Id$ and for $n \in \mathbb{N}$,*

$$\mathcal{A}(x, n) = \mathcal{A}_x^n = A(f^{n-1}x) \circ \cdots \circ A(x) \quad and \quad \mathcal{A}(x, -n) = \mathcal{A}_x^{-n} = (\mathcal{A}_{f^{-n}x}^n)^{-1}.$$

Clearly, $\mathcal{A}$ satisfies the cocycle equation $\mathcal{A}_x^{n+k} = \mathcal{A}_{f^k x}^n \circ \mathcal{A}_x^k$.

We say that the cocycle $\mathcal{A}$ is β-Hölder if its generator A is Hölder continuous with exponent $0 < \beta \leq 1$ with respect to the metric d, i.e. there exists $K > 0$ such that

$$d(A(x), A(y)) \leq K\, dist(x, y)^\beta \quad for\ all\ x, y \in X.$$

Hölder continuity is needed to develop a theory even for scalar cocycles. On the other hand, higher regularity is rare for cocycles given by restrictions of the differential of an Anosov diffeomorphism to the stable and unstable subbundles. Additionally, symbolic dynamical systems have a natural Hölder structure, but not a smooth one.

Definition 1.2. *The* quasiconformal distortion *of a cocycle $\mathcal{A}$ is the function*

$$Q_{\mathcal{A}}(x, n) = \|\mathcal{A}_x^n\| \cdot \|(\mathcal{A}_x^n)^{-1}\|, \quad x \in X\ and\ n \in \mathbb{Z}.$$

The quasiconformal distortion is a measure of non-conformality of the cocycle. If $Q_{\mathcal{A}}(x, n) \leq K$ for all x and n, the cocycle is said to be uniformly quasiconformal, and if $Q_{\mathcal{A}}(x, n) = 1$ for all x and n, it is said to be conformal.

Next we define fiber bunching of a cocycle. This condition means that non-conformality of the cocycle is dominated, in a sense, by the contraction and expansion in the base given by the functions ν and $\hat{\nu}$ in (2.1) and (2.2). In particular, bounded, conformal, and uniformly quasiconformal cocycles are fiber bunched.

Definition 1.3. *A β-Hölder cocycle $\mathcal{A}$ over a hyperbolic system (X, f) as in Section 2 is* fiber bunched *if there exist numbers $\theta < 1$ and L such that for all $x \in X$ and $n \in \mathbb{N}$,*

$$(1.2) \qquad Q_{\mathcal{A}}(x, n) \cdot (\nu_x^n)^{\beta} < L\,\theta^n \quad and \quad Q_{\mathcal{A}}(x, -n) \cdot (\hat{\nu}_x^{-n})^{\beta} < L\,\theta^n,$$

where $\nu_x^n = \nu(f^{n-1}x) \cdots \nu(x)$ and $\hat{\nu}_x^{-n} = (\hat{\nu}(f^{-n}x))^{-1} \cdots (\hat{\nu}(f^{-1}x))^{-1}$.

This condition guarantees convergence of certain iterates of the cocycle along the stable and unstable leaves, and it plays an important role in the study of non-commutative cocycles. We use the weakest, "pointwise", version of fiber bunching with non-constant estimates of expansion and contraction in the base.

Standing Assumptions 1.4. *In the results below, (X, f) is any of the transitive hyperbolic systems described in Section 2, and $\mathcal{A}$ and $\mathcal{B}$ are β-Hölder Banach cocycles over f.*

The next proposition allows us to obtain fiber bunching of a cocycle $\mathcal{A}$ from fiber bunching of its periodic data.

Proposition 1.5. *Suppose that for a cocycle $\mathcal{A}$ there exist numbers $\tilde{\theta} < 1$ and $\tilde{L}$ such that whenever $f^k p = p$, $k \in \mathbb{N}$, we have,*

$$(1.3) \qquad Q_{\mathcal{A}}(p, k) \cdot (\nu_p^k)^{\beta} < \tilde{L}\,\tilde{\theta}^k \quad and \quad Q_{\mathcal{A}}(p, -k) \cdot (\hat{\nu}_p^{-k})^{\beta} < \tilde{L}\,\tilde{\theta}^k.$$

Then $\mathcal{A}$ is fiber bunched.

It follows that a cocycle with periodic data conjugate to that of a fiber bunched cocycle via a bounded conjugacy is also fiber bunched.

Definition 1.6. *We say that Banach cocycles $\mathcal{A}$ and $\mathcal{B}$ have* conjugate periodic data *if for every periodic point $p = f^k p$ there exists $C(p) \in GL(V)$ such that*

$$\mathcal{B}_p^k = C(p) \circ \mathcal{A}_p^k \circ C(p)^{-1}.$$

Corollary 1.7. *Suppose that $\mathcal{A}$ is fiber bunched and $\mathcal{B}$ has conjugate periodic data with bounded conjugacy, i.e. $\max\{\|C(p)\|, \|C(p)^{-1}\|\} \leq M$, where M is a constant independent of p. Then $\mathcal{B}$ is also fiber bunched.*

A natural equivalence for cocycles is cohomology, i.e. existence of a conjugacy, which can be considered in various regularity classes.

Definition 1.8. *Cocycles $\mathcal{A}$ and $\mathcal{B}$ are (measurably, continuously)* cohomologous *if there exists a (measurable, continuous) function $C : X \to GL(V)$, called a* conjugacy *or a* transfer map *between $\mathcal{A}$ and $\mathcal{B}$, such that*

$$(1.4) \qquad \mathcal{A}_x^n = C(f^n x) \circ \mathcal{B}_x^n \circ C(x)^{-1} \quad for\ all\ x \in X\ and\ n \in \mathbb{Z}.$$

Clearly, if two cocycles are continuously cohomologous, then they have conjugate periodic data. The converse is not true in general even when V is two-dimensional and $C(p)$ is bounded [S13]. If $C(p)$ is Hölder, conjugating $\mathcal{B}$ by the extension of C reduces the problem to the case of equal periodic data, i.e. $\mathcal{A}_p^n = \mathcal{B}_p^n$. Positive results

for equal periodic data, as well as some results for conjugate data, were established by W. Parry [Pa99] for compact G and, somewhat more generally, by K. Schmidt [Sch99] for cocycles with "bounded distortion". First results outside this setting were obtained in [S13] for certain types of $GL(2, \mathbb{R})$-valued cocycles. In [S15] we considered $GL(n, \mathbb{R})$-valued cocycles over hyperbolic systems. We showed that if a cocycle $\mathcal{A}$ is fiber bunched and $\mathcal{B}$ has equal periodic data, then $\mathcal{A}$ and $\mathcal{B}$ are Hölder continuously cohomologous. Moreover, we obtained Hölder cohomology under the assumption that $\mathcal{A}$ is fiber bunched, $\mathcal{B}$ has conjugate periodic data and the conjugacy $C(p)$ is β-Hölder continuous at a fixed point of f. In the following theorem we extend the results to Banach cocycles and remove the assumption that f has a fixed point.

Theorem 1.9. *Suppose that a cocycle $\mathcal{A}$ is fiber bunched, $\mathcal{B}$ has conjugate periodic data, and $C(p)$ is β-Hölder continuous at a periodic point p_0, i.e. there exists a constant c such that $d(C(p), C(p_0)) \leq c \, dist(p, p_0)^{\beta}$ for every periodic point p.*

Then there exists a unique β-Hölder continuous conjugacy $\bar{C}$ between $\mathcal{A}$ and $\mathcal{B}$ such that $\bar{C}(p_0) = C(p_0)$.

We do not assume fiber bunching for $\mathcal{B}$ as it follows immediately from Corollary 1.7.

We note that $\bar{C}(p)$ does not necessarily coincide with $C(p)$ for $p \neq p_0$. For example, let $\mathcal{B} \equiv \mathrm{Id}$ and let $\mathcal{A}_x = \bar{C}(fx) \circ \bar{C}(x)^{-1} = \bar{C}(fx) \circ \mathcal{B}_x \circ \bar{C}(x)^{-1}$, where C is a Hölder continuous function with $\bar{C}(p_0) = \mathrm{Id}$. Then $\mathcal{A}_p^n = \mathcal{B}_p^n = \mathrm{Id}$ whenever $p = f^n p$, and so we can take $C(p) = \mathrm{Id}$ for each p.

Corollary 1.10. *Suppose that a cocycle $\mathcal{A}$ is fiber bunched and a cocycle $\mathcal{B}$ has equal periodic data, i.e. $\mathcal{A}_p^k = \mathcal{B}_p^k$ whenever $p = f^k p$. Then the cocycles are β-Hölder continuously cohomologous.*

In particular, if $\mathcal{B}$ is a cocycle with $\mathcal{B}_p^k = \mathrm{Id}$ whenever $p = f^k p$, then $\mathcal{B}$ is β-Hölder continuously cohomologous to the identity cocycle.

The second part of this corollary was recently obtained by G. Grabarnik and M. Guysinsky for cocycles with values in Banach algebras [GG].

Next we consider the question whether a measurable conjugacy between two cocycles is continuous, i.e. coincides with a continuous conjugacy on a set of full measure. Measurability is understood with respect to a suitable measure, for example the measure of maximal entropy or the invariant volume. This problem was also first considered in the case when one of the cocycles is the identity. The first result beyond this case was obtained by K. Schmidt [Sch99] for two cocycles with "bounded distortion", the prime examples being uniformly bounded $GL(n, \mathbb{R})$-valued cocycles and ones with values in compact groups. A counterexample by M. Pollicott and C. P. Walkden [PWa01] showed that additional assumptions are needed for a positive answer in more general context: they constructed $GL(2, \mathbb{R})$-valued cocycles which are measurably but not continuously cohomologous. Moreover both cocycles can be made arbitrarily close to the identity, and in particular fiber bunched. In [S15] we showed

that if $\mathcal{A}$ is fiber bunched and $\mathcal{B}$ is uniformly quasiconformal, i.e. $Q_{\mathcal{B}}(x,n) \leq K$ for all x and n, then any measurable conjugacy between $\mathcal{A}$ and $\mathcal{B}$ is β-Hölder continuous.

The following theorem extends the finite-dimensional results to the Banach setting. One of the difficulties here is that the space $(GL(V), d)$ is not separable even if V is. To use the tools of the theory of measurable functions, such as Lusin's theorem, we work with the strong operator topology, i.e. the topology of pointwise convergence. We assume that $\mathcal{B}$ takes values in a precompact set, which for a finite dimensional V is equivalent to uniform boundedness of $\mathcal{B}$ in $(GL(V), d)$.

Theorem 1.11. *Suppose that the Banach space V is separable, a cocycle $\mathcal{A}$ is fiber bunched and a cocycle $\mathcal{B}$ takes values in a subset of $GL(V)$ that is precompact in the topology of pointwise convergence. Let μ be an ergodic invariant measure with full support and local product structure. Then any μ-measurable conjugacy between $\mathcal{A}$ and $\mathcal{B}$ coincides with a β-Hölder continuous conjugacy on a set of full measure.*

The μ-measurability of the conjugacy means that the preimage of each Borel set in $GL(V)$ is μ-measurable and the conjugacy equation (1.4) holds μ-almost everywhere. We note that Borel σ-algebra in $GL(V)$ is the same for the metric d and for the strong operator topology. A measure has local product structure if it is locally equivalent to the product of its conditional measures on the local stable and unstable manifolds. Examples of ergodic measures with full support and local product structure include the measure of maximal entropy, more generally Gibbs (equilibrium) measures of Hölder continuous potentials, and the invariant volume if it exists.

2. HYPERBOLIC SYSTEMS IN THE BASE

Transitive Anosov diffeomorphisms. A diffeomorphism f of a compact connected manifold X is called *Anosov* if there exist a splitting of the tangent bundle TX into a direct sum of two Df-invariant continuous subbundles E^s and E^u, a Riemannian metric on X, and continuous functions ν and $\hat{\nu}$ such that

$$(2.1) \qquad \|Df_x(v^s)\| < \nu(x) < 1 < \hat{\nu}(x) < \|Df_x(v^u)\|$$

for any $x \in X$ and unit vectors $v^s \in E^s(x)$ and $v^u \in E^u(x)$. The sub-bundles E^s and E^u are called stable and unstable. They are tangent to the stable and unstable foliations W^s and W^u respectively.

We define the *local stable manifold* of x, $W^s_{\text{loc}}(x)$, as a ball centered at x of radius ρ in the intrinsic metric of $W^s(x)$. We choose ρ sufficiently small so that for every $x \in X$ we have $\|Df_y\| < \nu(x)$ for all y in $W^s_{\text{loc}}(x)$ and so that $W^s_{\text{loc}}(x) \cap W^u_{\text{loc}}(z)$ consists of a single point for any sufficiently close x and z in X. The second property is called the local product structure of the foliations. Local unstable manifolds are defined similarly. It follows that for all $n \in \mathbb{N}$,

$$(2.2) \qquad \begin{aligned} &\text{dist}(f^n x, f^n y) < \nu_x^n \cdot \text{dist}(x, y) \quad \text{for all } x \in X \text{ and } y \in W^s_{\text{loc}}(x), \\ &\text{dist}(f^{-n} x, f^{-n} y) < \hat{\nu}_x^{-n} \cdot \text{dist}(x, y) \quad \text{for all } x \in X \text{ and } y \in W^u_{\text{loc}}(x). \end{aligned}$$

A diffeomorphism f is *(topologically) transitive* if there is a point x in X with dense orbit. All known examples of Anosov diffeomorphisms have this property.

Topologically mixing diffeomorphisms of locally maximal hyperbolic sets.
More generally, let f be a diffeomorphism of a manifold $\mathcal{M}$. A compact f-invariant set $X \subseteq \mathcal{M}$ is called *hyperbolic* if there exist a continuous Df-invariant splitting $T_X\mathcal{M} = E^s \oplus E^u$, and a Riemannian metric and continuous functions ν, $\hat{\nu}$ on an open set $U \supseteq X$ such that (2.1) holds for all $x \in X$. The set X is called *locally maximal* if $X = \bigcap_{n \in \mathbb{Z}} f^{-n}(U)$ for some open set $U \supseteq X$.

Mixing subshifts of finite type. Let M be $k \times k$ matrix with entries from $\{0, 1\}$ such that all entries of M^N are positive for some N. Let

$$X = \{\, x = (x_n)_{n \in \mathbb{Z}} \; : \; 1 \le x_n \le k \;\; \text{and} \;\; M_{x_n, x_{n+1}} = 1 \;\; \text{for every} \;\; n \in \mathbb{Z} \,\}.$$

The shift map $f : X \to X$ is defined by $(fx)_n = x_{n+1}$. The system (X, f) is called a *mixing subshift of finite type*. We fix $\nu \in (0, 1)$ and consider the metric

$$\mathrm{dist}(x, y) = d_\nu(x, y) = \nu^{n(x,y)}, \quad \text{where} \;\; n(x, y) = \min\{\, |i| \; : \; x_i \ne y_i \,\}.$$

The following sets play the role of the local stable and unstable manifolds of x:

$$W^s_{\mathrm{loc}}(x) = \{\, y \mid x_i = y_i, \;\; i \ge 0 \,\}, \quad W^u_{\mathrm{loc}}(x) = \{\, y \mid x_i = y_i, \;\; i \le 0 \,\}$$

Indeed, for all $x \in X$ and $n \in \mathbb{N}$,

$$\mathrm{dist}(f^n x, f^n y) = \nu^n \cdot \mathrm{dist}(x, y) \quad \text{for all } y \in W^s_{\mathrm{loc}}(x),$$

$$\mathrm{dist}(f^{-n} x, f^{-n} y) = \nu^n \cdot \mathrm{dist}(x, y) \quad \text{for all } y \in W^u_{\mathrm{loc}}(x),$$

and for any $x, z \in X$ with $\mathrm{dist}(x, z) < 1$ the intersection of $W^s_{\mathrm{loc}}(x)$ and $W^u_{\mathrm{loc}}(z)$ consists of a single point. Thus, in this case we can take $\nu(x) = \nu$ and $\hat{\nu}(x) = \nu^{-1}$.

3. Proofs of Proposition 1.5 and Corollary 1.7

3.1. **Proof of Proposition 1.5.** We consider the sequence of real-valued functions

$$(3.1) \qquad a_n(x) = \log\left(Q_{\mathcal{A}}(x, n) \cdot (\nu^n_x)^\beta\right) = \log \|\mathcal{A}^n_x\| + \log \|(\mathcal{A}^n_x)^{-1}\| + \log(\nu^n_x)^\beta.$$

It is easy to verify that this sequence of functions is *subadditive*, i.e.

$$a_{n+k}(x) \le a_k(x) + a_n(f^k x) \quad \text{for all } x \in X \text{ and } n, k \in \mathbb{N}.$$

We recall some results on subadditive sequences. Let f be a homeomorphism of a compact metric space X and let a_n be a subadditive sequence of continuous functions from X to $\mathbb{R}$. For an ergodic f-invariant Borel probability measure μ on X, let $a_n(\mu) = \int_X a_n(x)\, d\mu$. The sequence of numbers $a_n(\mu)$ is subadditive, i.e. $a_{n+k}(\mu) \le a_n(\mu) + a_k(\mu)$, and it is well known that

$$\lim_{n \to \infty} a_n(\mu)/n = \inf\{\, a_n(\mu)/n \; : \; n \in \mathbb{N}\} =: \chi(a, \mu).$$

By the Subadditive Ergodic Theorem,

$$\lim_{n \to \infty} a_n(x)/n = \chi(a, \mu) \quad \text{for } \mu\text{-almost all } x \in X.$$

Lemma 3.1. [KS13, Proposition 4.9] *Let f be a homeomorphism of a compact metric space X and let $a_n : X \to \mathbb{R}$ be a subadditive sequence of continuous functions.*

If $\chi(a, \mu) < 0$ for every ergodic invariant Borel probability measure μ for f, then there exists N such that $a_N(x) < 0$ for all $x \in X$.

We will show that the assumption of the lemma is satisfied for the sequence $a_n(x)$ given by (3.1). We observe that for this sequence, $\chi(a, \mu)$ can be written in terms of Lyapunov exponents of the cocycles $\mathcal{A}$ and ν^β.

Definition 3.2. *Let μ be an ergodic f-invariant Borel probability measure on X. The* upper and lower Lyapunov exponents *of $\mathcal{A}$ with respect to μ are*

$$\lambda_+(\mathcal{A}, \mu) = \lim_{n \to \infty} \frac{1}{n} \log \|\mathcal{A}_x^n\| \quad and \quad \lambda_-(\mathcal{A}, \mu) = \lim_{n \to \infty} \frac{1}{n} \log \|(\mathcal{A}_x^n)^{-1}\|^{-1}.$$

By the Subadditive Ergodic Theorem, each of these limits exists and is the same μ almost everywhere. Now for the sequence in (3.1) for μ almost every x we have:

$$(3.2) \qquad \chi(a, \mu) = \lim_{n \to \infty} a_n(x)/n = \lambda_+(\mathcal{A}, \mu) - \lambda_-(\mathcal{A}, \mu) + \lambda(\nu^\beta, \mu),$$

where $\lambda(\nu^\beta, \mu) = \lambda_+(\nu^\beta, \mu) = \lambda_-(\nu^\beta, \mu)$ for the scalar cocycle ν^β.

The next theorem gives an approximation of the upper and lower Lyapunov exponents of a cocycle in terms of its periodic data. It is Theorem 1.4 and Remark 1.5 in [KS16] stated for our setting.

Theorem 3.3. *Let (X, f) be a hyperbolic system, let μ be an ergodic f-invariant Borel probability measure on X, and let $\mathcal{A}$ be a Hölder continuous Banach cocycle over f. Then for each $\epsilon > 0$ there exists a periodic point $p = f^k p$ in X such that*

$$(3.3) \qquad \left| \lambda_+(\mathcal{A}, \mu) - \frac{1}{k} \log \|\mathcal{A}_p^k\| \right| < \epsilon \quad and \quad \left| \lambda_-(\mathcal{A}, \mu) - \frac{1}{k} \log \|(\mathcal{A}_p^k)^{-1}\|^{-1} \right| < \epsilon.$$

Moreover, given finitely many Hölder continuous Banach cocycles over f and $K \in \mathbb{N}$, there exists a periodic point $p = f^k p$ with $k > K$ which gives simultaneous approximation (3.3) for all the cocycles.

Let $\tilde{L}$ and $\tilde{\theta} < 1$ be as in the assumption (1.3). We choose $K \in \mathbb{N}$ such that $-3\epsilon := (\log \tilde{L})/K + \log \tilde{\theta} < 0$. Then for each point $p = f^k p$ with $k \geq K$ we have

$$\frac{1}{k} \log \left(\|\mathcal{A}_p^k\| \cdot \|(\mathcal{A}_p^k)^{-1}\| \cdot (\nu_p^k)^\beta \right) \leq \frac{1}{k} \log(\tilde{L} \tilde{\theta}^k) = \frac{1}{k} \log \tilde{L} + \log \tilde{\theta} < -3\epsilon.$$

By Theorem 3.3 there exists a periodic point $p = f^k p$ with $k > K$ such that (3.3) holds for $\mathcal{A}$, and also for the scalar cocycle ν^β we have that $\lambda(\nu^\beta, \mu_p) = \frac{1}{k} \log(\nu_p^k)^\beta$ is ϵ-close to $\lambda(\nu^\beta, \mu)$. Then by (3.2) we have

$$\left| \chi(a, \mu) - \frac{1}{k} \log \left(\|\mathcal{A}_p^k\| \cdot \|(\mathcal{A}_p^k)^{-1}\| \cdot (\nu_p^k)^\beta \right) \right| < 3\epsilon$$

and hence $\chi(a, \mu) < 0$. Then by Lemma 3.1 there exists N such that $a_N(x) < 0$ for all x, and so $Q_{\mathcal{A}}(x, N) \cdot (\nu_x^N)^\beta < 1$ for all $x \in X$. By continuity, there exists $\theta < 1$ such that the left hand side is smaller than θ for all x. Writing $n \in \mathbb{N}$ as $n = mN + r$, where $m \in \mathbb{N} \cup \{0\}$ and $0 \le r < N$, we obtain

$$Q_{\mathcal{A}}(x, n) \cdot (\nu_x^n)^\beta \le Q_{\mathcal{A}}(x, mN) \cdot (\nu_x^{mN})^\beta \cdot Q_{\mathcal{A}}(f^{mN} x, r) \cdot (\nu_{f^{mN} x}^r)^\beta \le L \, \theta^n,$$

where $L = \max \left\{ Q_{\mathcal{A}}(x, r) \, (\nu_x^r)^\beta \, \theta^{-N} : \ x \in X, \ 0 \le r < N \right\}$. The inequality with $\hat{\nu}$ is obtained similarly, and we conclude that the cocycle $\mathcal{A}$ is fiber bunched. $\qquad\square$

3.2. **Proof of Corollary 1.7.** Since the cocycle $\mathcal{A}$ is fiber bunched, there exist numbers L and $\theta < 1$ such that $Q_{\mathcal{A}}(x, n) \cdot (\nu_x^n)^\beta < L \, \theta^n$ for all $x \in X$ and $n \in \mathbb{N}$. It follows that whenever $p = f^k p$,

$$Q_{\mathcal{B}}(p, k) \cdot (\nu_p^k)^\beta \le \|C(p)\| \cdot Q_{\mathcal{A}}(p, k) \cdot \|C(p)^{-1}\| \cdot (\nu_p^k)^\beta \le M^2 L \, \theta^k.$$

Hence by Proposition 1.5 the cocycle $\mathcal{B}$ is fiber bunched. $\qquad\square$.

4. Proof of Theorem 1.9

4.1. **Cocycles over systems with a fixed point.** The following result was established in [S15] for $GL(d, \mathbb{R})$-valued cocycles.

Theorem 4.1. [S15, Theorem 2.4] *Suppose that $\mathcal{A}$ is fiber bunched and $\mathcal{B}$ has conjugate periodic data. In addition, suppose that f has a fixed point p_0 and the conjugacy $C(p)$ is β-Hölder continuous at p_0, i.e. $d(C(p), C(p_0)) \le c \, dist(p, p_0)^\beta$ for every periodic point p. Then there exists a unique β-Hölder continuous conjugacy $\bar{C}$ between $\mathcal{A}$ and $\mathcal{B}$ such that $\bar{C}(p_0) = C(p_0)$.*

For Banach cocycles, the proof holds without modifications, except for using Corollary 1.7 instead of Proposition 2.3 in [S15] to obtain fiber bunching of $\mathcal{B}$.

We give an outline of the proof of this theorem since we will refer to it later. We consider the cocycle $\tilde{\mathcal{B}} = C(p_0) \circ \mathcal{B} \circ C(p_0)^{-1}$ and the function $\tilde{C}(p) = C(p)C(p_0)^{-1}$, so that $\tilde{\mathcal{B}}_{p_0} = \mathcal{A}_{p_0}$ and $\tilde{C}(p_0) = \mathrm{Id}$. If $\tilde{C}(x)$ is a conjugacy between $\mathcal{A}$ and $\tilde{\mathcal{B}}$ with $\tilde{C}(p_0) = \mathrm{Id}$, then $\bar{C}(x) = \tilde{C}(x)C(p_0)$ is a conjugacy between $\mathcal{A}$ and $\mathcal{B}$ with $\bar{C}(p_0) = C(p_0)$. Thus it suffices to consider the case when $\mathcal{B}_{p_0} = \mathcal{A}_{p_0}$ and $C(p_0) = \mathrm{Id}$. First the conjugacy $\bar{C}(x)$ is constructed along the stable and unstable manifolds of p_0 using the stable and unstable holonomies (see Section 4.2 below):

$$(4.1) \qquad \begin{aligned} \bar{C}^s(x) &= H_{p_0, x}^{\mathcal{A}, s} \circ H_{x, p_0}^{\mathcal{B}, s} \quad \text{for } x \in W^s(p_0), \\ \bar{C}^u(x) &= H_{p_0, x}^{\mathcal{A}, u} \circ H_{x, p_0}^{\mathcal{B}, u} \quad \text{for } x \in W^u(p_0). \end{aligned}$$

For a homoclinic point x for p_0, i.e. $x \in W^s(p_0) \cap W^u(p_0)$, it is shown that

$$\bar{C}^s(x) = \bar{C}^u(x) \stackrel{def}{=} \bar{C}(x).$$

Thus we obtain $\bar{C}$ on the set of homoclinic points of p_0, which is dense in X. The function $\bar{C}$ is β-Hölder continuous on this set, and hence it can be extended to X. This completes the outline.

4.2. **Holonomies.** An important role in the proof of Theorem 4.1, as well as in the proof of Theorem 1.9, is played by holonomies of cocycles. This terminology for certain limits of iterates of a linear cocycle was introduced in [V08] and the holonomies were further studied in [ASV13, KS13, S15]. The result below gives existence of holonomies under the weakest fiber bunching assumption (1.2).

Let $\mathcal{V} = X \times V$ be a trivial vector bundle over X. For a Banach cocycle $\mathcal{A}$, we view $\mathcal{A}_x$ as a linear map from $\mathcal{V}_x$, the fiber at x, to $\mathcal{V}_{fx}$, so $\mathcal{A}_x^n : \mathcal{V}_x \to \mathcal{V}_{f^n x}$ and $\mathcal{A}_x^{-n} : \mathcal{V}_x \to \mathcal{V}_{f^{-n} x}$.

Proposition 4.2. [S15, Proposition 4.4] *Suppose that a cocycle $\mathcal{A}$ is fiber bunched. Then for every $x \in X$ and $y \in W^s(x)$ the limit*

$$(4.2) \qquad H_{x,y}^{\mathcal{A},s} = \lim_{n \to \infty} (\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n,$$

exists and satisfies

(H1) $H_{x,y}^{\mathcal{A},s}$ *is a linear map from $\mathcal{V}_x$ to $\mathcal{V}_y$;*

(H2) $H_{x,x}^{\mathcal{A},s} = Id$ *and* $H_{y,z}^{\mathcal{A},s} \circ H_{x,y}^{\mathcal{A},s} = H_{x,z}^{\mathcal{A},s}$, *which implies* $(H_{x,y}^{\mathcal{A},s})^{-1} = H_{y,x}^{\mathcal{A},s}$;

(H3) $H_{x,y}^{\mathcal{A},s} = (\mathcal{A}_y^n)^{-1} \circ H_{f^n x, f^n y}^{\mathcal{A},s} \circ \mathcal{A}_x^n$ *for all $n \in \mathbb{N}$;*

(H4) $\|H_{x,y}^{\mathcal{A},s} - Id\| \le c\, dist(x,y)^\beta$, *where c is independent of x and $y \in W_{loc}^s(x)$.*

The continuous map $H^{\mathcal{A},s} : (x,y) \mapsto H_{x,y}^{\mathcal{A},s}$, where $x \in X$, $y \in W^s(x)$, is called the (standard) *stable holonomy* for $\mathcal{A}$. The unstable holonomy $H^{\mathcal{A},u}$ is defined similarly:

$$H_{x,y}^{\mathcal{A},u} = \lim_{n \to \infty} \left((\mathcal{A}_y^{-n})^{-1} \circ (\mathcal{A}_x^{-n}) \right) = \lim_{n \to \infty} \left(\mathcal{A}_{f^{-n}y}^n \circ (\mathcal{A}_{f^{-n}x}^n)^{-1} \right), \text{ where } y \in W^u(x).$$

It satisfies (H1, 2, 4) and

(H3′) $H_{x,y}^{\mathcal{A},u} = (\mathcal{A}_y^{-n})^{-1} \circ H_{f^{-n}x, f^{-n}y}^{\mathcal{A},u} \circ \mathcal{A}_x^{-n}$ *for all $n \in \mathbb{N}$.*

4.3. **Removing the fixed point assumption.** Now we obtain the Hölder continuous conjugacy between the cocycles assuming Hölder continuity of $C(p)$ at a periodic point p_0. As was explained before, we can assume that $\mathcal{A}_{p_0} = \mathcal{B}_{p_0}$ and $C(p_0) = Id$. Let k be a period of the point p_0. Then p_0 is a fixed point for f^k and so Theorem 4.1 gives a unique Hölder conjugacy $\bar{C}$ between the iterated cocycles $\mathcal{A}^k$ and $\mathcal{B}^k$ over f^k with $\bar{C}(p_0) = Id$. We will show that $\bar{C}$ is also a conjugacy between $\mathcal{A}$ and $\mathcal{B}$ following the approach of [BK16, Sch99].

To simplify the notations we write p for p_0 and C for $\bar{C}$. We consider a point $x \in W^s(p) \cap W^u(f^{k-1}p)$, so that $fx \in W^u(p)$. We will show that

$$(4.3) \qquad \mathcal{A}_x = C(fx) \circ \mathcal{B}_x \circ C(x)^{-1}, \quad \text{i.e.} \quad C(x) = \mathcal{A}_x^{-1} \circ C(fx) \circ \mathcal{B}_x.$$

Since such points x are dense in X and all the functions are continuous, it follows that the equation holds for all $x \in X$.

Now we prove (4.3). Using equations (4.1) we obtain

$$C(x) = C^s(x) = H_{p,\,x}^{A^k,s} \circ H_{x,\,p}^{B^k,s} =$$
$$= \lim_{n\to\infty} (\mathcal{A}_x^{nk})^{-1} \circ \mathcal{A}_p^{nk} \circ (\mathcal{B}_p^{nk})^{-1} \circ \mathcal{B}_x^{nk} = \lim_{n\to\infty} (\mathcal{A}_x^{nk})^{-1} \circ \mathcal{B}_x^{nk}, \quad \text{and}$$

$$C(fx) = C^u(fx) = H_{p,\,fx}^{A^k,u} \circ H_{fx,\,p}^{B^k,u} = \lim_{m\to\infty} \mathcal{A}_{f^{-mk}fx}^{mk} \circ (\mathcal{B}_{f^{-mk}fx}^{mk})^{-1}$$
$$= \lim_{m\to\infty} \mathcal{A}_{f^{-mk+1}x}^{mk} \circ (\mathcal{B}_{f^{-mk+1}x}^{mk})^{-1}.$$

Since $\mathcal{A}_x^{-1} \circ C(fx) \circ \mathcal{B}_x$ equals

$$\lim_{m\to\infty} \mathcal{A}_x^{-1} \circ \mathcal{A}_{f^{-mk+1}x}^{mk} \circ (\mathcal{B}_{f^{-mk+1}x}^{mk})^{-1} \circ \mathcal{B}_x = \lim_{m\to\infty} \mathcal{A}_{f^{-mk+1}x}^{mk-1} \circ (\mathcal{B}_{f^{-mk+1}x}^{mk-1})^{-1},$$

we need to show that

$$\lim_{n\to\infty} (\mathcal{A}_x^{nk})^{-1} \circ \mathcal{B}_x^{nk} = \lim_{m\to\infty} \mathcal{A}_{f^{-mk+1}x}^{mk-1} \circ (\mathcal{B}_{f^{-mk+1}x}^{mk-1})^{-1}.$$

As both limits exist, so does the following limit and it suffices to prove the equality

$$\lim_{m,n\to\infty} (\mathcal{A}_x^{nk})^{-1} \circ \mathcal{B}_x^{nk} \circ \mathcal{B}_{f^{-mk+1}x}^{mk-1} \circ (\mathcal{A}_{f^{-mk+1}x}^{mk-1})^{-1} = \mathrm{Id}.$$

Since $x \in W^s(p)$ and $fx \in W^u(p)$,

$$f^{nk}x \to p \text{ as } n \to \infty \quad \text{and} \quad f^{-mk+1}x = f^{-mk}fx \to p \text{ as } m \to \infty.$$

Moreover, by (2.1) there is a constant $c_1(x)$ such that for all $n,m \in \mathbb{N}$,

$$\mathrm{dist}(f^{nk}x,\,p) < \nu_x^{nk} \cdot c_1(x)\,\mathrm{dist}_{W^s(p)}(x,p) =: c_2(x)\,\nu_x^{nk} \quad \text{and}$$
$$\mathrm{dist}(f^{-mk+1}x,\,p) < \hat{\nu}_x^{-mk+1} \cdot c_1(x)\,\mathrm{dist}_{W^u(p)}(x,p) =: c_3(x)\,\hat{\nu}_x^{-mk+1}.$$

Let δ_0 be as in Anosov Closing Lemma [KtH, Theorem 6.4.15]. We take $\delta < \delta_0$ and let m and n be the smallest positive integers such that both distances above are less than $\delta/2$. Then we have

(4.4) $\qquad \mathrm{dist}(f^{nk}x,\,f^{-mk+1}x) < \delta \quad \text{and} \quad \delta < c_4(x)\min\{\nu_x^{nk}, \hat{\nu}_x^{-mk+1}\}.$

Applying the lemma to the orbit segment $\{f^i x : \ i = -mk+1, \ldots, nk\}$, we obtain a periodic point $q = f^{(m+n)k-1}q$ such that

$$\mathrm{dist}(f^i x,\, f^i q) \leq L\delta \quad \text{for } i = -mk+1,\,\ldots,nk.$$

Let z be the intersection point of $W_{\mathrm{loc}}^s(q)$ and $W_{\mathrm{loc}}^u(x)$. Then by the local product structure,

(4.5) $\quad \mathrm{dist}(f^i z,\, f^i x) \leq c_5\delta \text{ and } \mathrm{dist}(f^i z,\, f^i q) \leq c_5\delta \quad \text{for } i = -mk+1, \ldots, nk.$

Since $z \in W^s(q)$, property (H3) of the holonomies yields

$$\mathcal{A}_z^{nk} = H_{f^{nk}q,\,f^{nk}z}^{A,s} \circ \mathcal{A}_q^{nk} \circ H_{z,\,q}^{A,s},$$

and since $f^{nk}z \in W^u(f^{nk}x)$ using property (H3$'$) we obtain

$$\mathcal{A}_x^{nk} = H_{f^{nk}z,\, f^{nk}x}^{\mathcal{A},u} \circ \mathcal{A}_z^{nk} \circ H_{x,z}^{\mathcal{A},u}.$$

Thus

$$\mathcal{A}_x^{nk} = H_{f^{nk}z,\, f^{nk}x}^{\mathcal{A},u} \circ H_{f^{nk}q,\, f^{nk}z}^{\mathcal{A},s} \circ \mathcal{A}_q^{nk} \circ H_{z,q}^{\mathcal{A},s} \circ H_{x,z}^{\mathcal{A},u}.$$

It follows from property (H4) that

$$H_{z,q}^{s,\mathcal{A}} = \mathrm{Id} + R_{z,q}^{s,\mathcal{A}}, \quad \text{where } \|R_{z,q}^{s,\mathcal{A}}\| \le c \operatorname{dist}(z,q)^\beta \le c_6 \delta^\beta.$$

Similar estimates hold for the other holonomies, as well as their inverses, due to (4.5). Thus we obtain

(4.6) $\quad (\mathcal{A}_x^{nk})^{-1} = (\mathrm{Id} + R_1) \circ (\mathcal{A}_q^{nk})^{-1} \circ (\mathrm{Id} + R_2), \quad \text{where } \|R_1\|, \|R_2\| \le c_6 \delta^\beta,$

and similarly,

(4.7) $\quad (\mathcal{A}_{f^{-mk+1}x}^{mk-1})^{-1} = (\mathrm{Id} + R_3) \circ (\mathcal{A}_{f^{-mk+1}q}^{mk-1})^{-1} \circ (\mathrm{Id} + R_4),$

where $\|R_3\|, \|R_4\| \le c_6 \delta^\beta$. Let $q' = f^{-mk+1}q = f^{nk}q$ and $x' = f^{-mk+1}x$. Then

(4.8) $\quad \mathcal{B}_x^{nk} \circ \mathcal{B}_{f^{-mk+1}x}^{mk-1} = \mathcal{B}_{x'}^{(m+n)k-1} = (\mathrm{Id} + R_5) \circ \mathcal{B}_{q'}^{(m+n)k-1} \circ (\mathrm{Id} + R_6),$

where $\|R_5\|, \|R_6\| \le c_6 \delta^\beta$. Since q' is a periodic point of period $(m+n)k - 1$, by the assumption there exists $C(q')$ such that

(4.9)
$$\mathcal{B}_{q'}^{(m+n)k-1} = C(q') \circ \mathcal{A}_{q'}^{(m+n)k-1} \circ C(q')^{-1}, \quad \text{where } C(q') = \mathrm{Id} + R_7 \text{ and}$$
$$C(q')^{-1} = \mathrm{Id} + R_8 \quad \text{with } \|R_7^n\|, \|R_8^n\| \le c_7' \operatorname{dist}(p, q')^\beta \le c_7 \delta^\beta.$$

It follows from (4.8) and (4.9) that

(4.10) $\quad \mathcal{B}_x^{nk} \circ \mathcal{B}_{f^{-mk+1}x}^{mk-1} = (\mathrm{Id} + R_9) \circ \mathcal{A}_q^{nk} \circ \mathcal{A}_{f^{-mk+1}q}^{mk-1} \circ (\mathrm{Id} + R_{10}),$

where $\|R_9\|, \|R_{10}\| \le c_8 \delta^\beta$.

Using (4.6), (4.7), and (4.10) and combining terms of the form $\mathrm{Id} + R_i$ we obtain

$(\mathcal{A}_x^{nk})^{-1} \circ \mathcal{B}_x^{nk} \circ \mathcal{B}_{f^{-mk+1}x}^{mk-1} \circ (\mathcal{A}_{f^{-mk+1}x}^{mk-1})^{-1} =$

$(\mathrm{Id} + R_1) \circ (\mathcal{A}_q^{nk})^{-1} \circ (\mathrm{Id} + R_{11}) \circ \mathcal{A}_q^{nk} \circ \mathcal{A}_{f^{-mk+1}q}^{mk-1} \circ (\mathrm{Id} + R_{12}) \circ (\mathcal{A}_{f^{-mk+1}q}^{mk-1})^{-1} \circ (\mathrm{Id} + R_4)$

$= \mathrm{Id} + (\mathcal{A}_q^{nk})^{-1} \circ R_{11} \circ \mathcal{A}_q^{nk} \circ \mathcal{A}_{f^{-mk+1}q}^{mk-1} \circ R_{12} \circ (\mathcal{A}_{f^{-mk+1}q}^{mk-1})^{-1}$

$\qquad + (\mathcal{A}_q^{nk})^{-1} \circ R_{11} \circ \mathcal{A}_q^{nk} + \mathcal{A}_{f^{-mk+1}q}^{mk-1} \circ R_{12} \circ (\mathcal{A}_{f^{-mk+1}q}^{mk-1})^{-1} + \text{ smaller terms.}$

Since $\|R_i\| \le c_9 \delta^\beta$ for each i, where δ satisfies (4.4), and the cocycle is fiber bunched, we have

$\|\mathcal{A}_q^{nk}\| \cdot \|(\mathcal{A}_q^{nk})^{-1}\| \cdot \|R_{11}\| \le Q_{\mathcal{A}}(q, nk) \cdot c \delta^\beta \le Q_{\mathcal{A}}(q, nk) \cdot c_{10}(\nu_x^{nk})^\beta = c_{10} L \theta^{nk} \quad \text{and}$

$\|\mathcal{A}_{f^{-mk+1}q}^{mk-1}\| \cdot \|(\mathcal{A}_{f^{-mk+1}q}^{mk-1})^{-1}\| \cdot \|R_{12}\| \le c_{10}(\hat{\nu}_x^{-mk+1})^\beta = c_{10} L \theta^{mk-1}.$

We conclude that in the operator norm topology,

$$\lim_{n,m\to\infty} (\mathcal{A}_x^{nk})^{-1} \circ \mathcal{B}_x^{nk} \circ \mathcal{B}_{f^{-mk+1}x}^{mk-1} \circ (\mathcal{A}_{f^{-mk+1}x}^{mk-1})^{-1} = \mathrm{Id},$$

and so (4.3) holds. □

5. Proof of Theorem 1.11

We consider the strong operator topology, i.e. the topology of pointwise convergence, on the space of linear operators $L(V)$. It is induced by the family of semi-norms $P_v(A) = \|A(v)\|$, where $v \in V$. This topology is weaker than the operator norm topology, but they generate the same Borel σ-algebra. Indeed, for any $A \in L(V)$ we have $\|A\| = \sup \|A(v_i)\| = \sup P_{v_n}(A)$, where the supremum is taken over a dense set $\{v_n : n \in \mathbb{N}\}$ in the unit ball in V. Since the functions P_{v_n} are continuous in the strong operator topology, it follows that $\|A\|$ is Borel measurable and thus the balls in the operator norm belong to the Borel σ-algebra of the strong operator topology.

Since V is separable, this topology is separable and metrizable on any set $G \subset L(V)$ that is bounded in norm. Indeed, let $\{v_n : n \in \mathbb{N}\}$ be a countable dense set in the unit ball in V. Then

$$(5.1) \qquad \bar{d}(A, B) = \sum_{n=1}^{\infty} \frac{\|A(v_n) - B(v_n)\|}{1 + \|A(v_n) - B(v_n)\|} \cdot 2^{-n}$$

is a distance on $L(V)$. The convergence in $\bar{d}$ is the pointwise convergence on the set $\{v_n : n \in \mathbb{N}\}$. It induces the strong operator topology on G since for a bounded sequence convergence on each v_n is equivalent to convergence on each $v \in V$. To show separability, we take the set $\{v_n\}$ in the definition of $\bar{d}$ to be linearly independent and consider a countable dense set $U = \{u_n : n \in \mathbb{N}\}$ in V. Then the set of all finite sequences $\{u_{n_1}, \ldots, u_{n_k}\}$ in U is countable and for each finite sequence we can take a bounded operator $B_{n_1,\ldots,n_k}$ such that

$$B_{n_1,\ldots,n_k}(v_i) = u_{n_i} \text{ for } i = 1, \ldots, k.$$

This can be done by extending the coordinate functionals x_i on $\mathrm{span}\{v_1, \ldots, v_k\}$ to V using Hahn-Banach Theorem and defining $B_{n_1,\ldots,n_k}(v) = \sum_{i=1}^{k} x_i(v)u_{n_i}$. The set of such operators is a countable dense set in $(L(V), \bar{d})$. Indeed, given $\epsilon > 0$ and $A \in L(V)$ we can fix a large k so that the "tail" of the series in (5.1) is small and then choose u_{n_i} sufficiently close to $A(v_i)$ so that

$$\|A(v_i) - B_{n_1,\ldots,n_k}(v_i)\| = \|A(v_i) - u_{n_i}\| \quad \text{is small for } i = 1, \ldots, k.$$

Thus the strong operator topology is separable and hence second countable on G.

The corresponding strong operator topology on $GL(V)$ is induced by the embedding $i : GL(V) \to L(V) \times L(V)$ given by $i(A) = (A, A^{-1})$. The convergence in this

topology is the pointwise convergence of operators and their inverses: a sequence A_n converges to A if for each $v \in V$,

$$\|A_n(v) - A(v)\| + \|A_n^{-1}(v) - A^{-1}(v)\| \to 0 \quad \text{as } n \to \infty.$$

It follows from the results for $L(V)$ that this topology is separable and metrizable by

$$\tilde{d}(A, B) = \bar{d}(A, B) + \bar{d}(A^{-1}, B^{-1})$$

on any set $G \subset GL(V)$ bounded with respect to the metric d given by (1.1).

By the Uniform Boundedness Principle, a sequence A_n that converges in strong operator topology on $GL(V)$ is bounded in the metric d, and hence any subset of $GL(V)$ that is compact in the strong operator topology is bounded in d. Since the cocycle $\mathcal{B}$ takes values in such a subset, $\mathcal{B}$ is uniformly bounded, and thus satisfies the fiber bunching condition (1.2).

Let C be a μ-measurable conjugacy between $\mathcal{A}$ and $\mathcal{B}$. First we show that C intertwines holonomies of $\mathcal{A}$ and $\mathcal{B}$ on a set of full measure, i.e. there exists a set $Y \subset X$ with $\mu(Y) = 1$ such that

(5.2) $\qquad H_{x,y}^{\mathcal{A},s} = C(y) \circ H_{x,y}^{\mathcal{B},s} \circ C(x)^{-1} \quad \text{for all } x, y \in Y \text{ such that } y \in W^s(x),$

and a similar statement holds for the unstable holonomies. Since

$$C(x) = (\mathcal{A}_x^n)^{-1} \circ C(f^n x) \circ \mathcal{B}_x^n \quad \text{and} \quad H_{x,y}^{\mathcal{A},s} = (\mathcal{A}_y^n)^{-1} \circ H_{f^n x, f^n y}^{\mathcal{A},s} \circ \mathcal{A}_x^n,$$

it suffices to prove that $H_{f^n x, f^n y}^{\mathcal{A},s} = C(f^n y) \circ H_{f^n x, f^n y}^{\mathcal{B},s} \circ C(f^n x)^{-1}$. Thus we can assume that y lies on the local stable manifold of x.

Let $x \in X$ and $y \in W_{\text{loc}}^s(x)$. Since $\mathcal{A}_x = C(fx) \circ \mathcal{B}_x \circ C(x)^{-1}$, we have

$$(\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n = C(y) \circ (\mathcal{B}_y^n)^{-1} \circ C(f^n y)^{-1} \circ C(f^n x) \circ \mathcal{B}_x^n \circ C(x)^{-1} =$$

(5.3) $\qquad = C(y) \circ (\mathcal{B}_y^n)^{-1} \circ (\text{Id} + r_n) \circ \mathcal{B}_x^n \circ C(x)^{-1} =$

$$= C(y) \circ (\mathcal{B}_y^n)^{-1} \circ \mathcal{B}_x^n \circ C(x)^{-1} + C(y) \circ (\mathcal{B}_y^n)^{-1} \circ r_n \circ \mathcal{B}_x^n \circ C(x)^{-1},$$

where

$$r_n = C(f^n y)^{-1} \circ C(f^n x) - \text{Id} = C(f^n y)^{-1} \circ (C(f^n x) - C(f^n y)).$$

Since C is μ-measurable, $\|C\|$ and $\|C^{-1}\|$ are measurable functions from X to $\mathbb{R}$ and hence there exists a compact set $S_1 \subset X$ with $\mu(S_1) > 3/4$ such that $\|C\|$ and $\|C^{-1}\|$ are bounded on S_1. Let $G \subset GL(V)$ be a d-bounded set that contains the values of C and C^{-1} on the set S_1. Then C restricted to S_1 is a μ-measurable function to the separable, and hence second countable, metric space $(G, \tilde{d})$. Hence by Lusin's theorem there exists a compact set $S \subset S_1$ with $\mu(S) > 1/2$ such that C is uniformly continuous on S.

Let Y be the set of points in X for which the frequency of visiting S equals $\mu(S) > 1/2$. By Birkhoff Ergodic Theorem, $\mu(Y) = 1$. If x and y are in Y, there exists a sequence $\{n_i\}$ such that $f^{n_i} x$ and $f^{n_i} y$ are in S for all i. Thus $\|C\|, \|C^{-1}\|$

are uniformly bounded on the set $\{f^{n_i}x,\ f^{n_i}y\}$ and $\|(C(f^{n_i}x) - C(f^{n_i}y))(v)\| \to 0$ for every $v \in V$. Hence

$$(5.4) \qquad \text{the sequence } \|r_{n_i}\| \text{ is bounded and } \|r_{n_i}(v)\| \to 0 \quad \text{for every } v \in V.$$

We rewrite the formula (5.3) as

$$(5.5) \qquad C(y)^{-1} \circ (\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n \circ C(x) - (\mathcal{B}_y^n)^{-1} \circ \mathcal{B}_x^n = (\mathcal{B}_y^n)^{-1} \circ r_n \circ \mathcal{B}_x^n.$$

By (4.2) the left hand side converges in the operator norm to

$$(5.6) \qquad C(y)^{-1} \circ H_{x,y}^{\mathcal{A},s} \circ C(x) - H_{x,y}^{\mathcal{B},s}.$$

Hence the right hand side of (5.5) has a limit in the operator norm. To prove that it equals 0, it suffices to show that for each $v \in V$, $((\mathcal{B}_y^n)^{-1} \circ r_n \circ \mathcal{B}_x^n)(v)$ tends to 0 along a subsequence.

Let $\{n_i\}$ be the sequence in (5.4). Since $\mathcal{B}$ takes values in a compact subset of $GL(V)$ with strong operator topology, there exists a subsequence $\{n_{i_j}\}$ of $\{n_i\}$ such that $(\mathcal{B}_x^{n_{i_j}})(v)$ converges for every v and, as we observed above, $(\mathcal{B}_y^n)^{-1}$ is uniformly bounded. Now it follows from (5.4) that

$$\left((\mathcal{B}_y^{n_{i_j}})^{-1} \circ r_{n_{i_j}} \circ \mathcal{B}_x^{n_{i_j}} \right)(v) \to 0 \quad \text{for every } v \in V.$$

We conclude that (5.6) equals 0 and so (5.2) follows. The statement for the unstable holonomies is proven similarly.

Now we establish Hölder continuity of C on a set of full measure. We consider a small open set U in X with the product structure of stable and unstable manifolds,

$$U = W_{\text{loc}}^s(x_0) \times W_{\text{loc}}^u(x_0) = \{W_{\text{loc}}^s(x) \cap W_{\text{loc}}^u(y)\ :\ x \in W_{\text{loc}}^s(x_0) \text{ and } y \in W_{\text{loc}}^u(x_0)\}.$$

We take a finite cover of X by such sets. It suffices to show that C is Hölder continuous on a full measure subset of each such set U.

Since the measure μ has local product structure, μ is equivalent to the product of conditional measures on $W_{\text{loc}}^s(x_0)$ and $W_{\text{loc}}^u(x_0)$, and hence for μ almost every local stable leaf in U, the set of points of Y on the leaf has full conditional measure. Let Y_U be the the set of points in $Y \cap U$ that lie on such leaves. Then Y_U has full measure in U. Since the holonomies of the unstable foliation are absolutely continuous with respect to the conditional measures, for any two points x and z in Y_U, there exists a point $y \in W_{\text{loc}}^s(x) \cap Y_U$ such that $y' = W_{\text{loc}}^u(y) \cap W_{\text{loc}}^s(z)$ is also in Y_U.

Now we show that C and C^{-1} are bounded on Y_U. We fix $x \in Y_U$ and for any $z \in Y_U$ we consider y and y' as above. Then by equation (5.2) and property (H4) of the holonomies we have

$$(5.7) \qquad C(y) = H_{x,y}^{\mathcal{A},s} \circ C(x) \circ (H_{x,y}^{\mathcal{B},s})^{-1} = (\text{Id} + R_{x,y}^{\mathcal{A},s}) \circ C(x) \circ (\text{Id} + R_{x,y}^{\mathcal{B},s}),$$

where $\|R_{x,y}^{\mathcal{A},s}\|$, $\|R_{x,y}^{\mathcal{B},s}\| \le c \operatorname{dist}(x,y)^{\beta}$. If the set U is sufficiently small, then the terms R are less than one in norm and it follows that $\|C(y)\| \le 4\|C(x)\|$. Also considering the pairs y, y' and y', z we conclude that $\|C(z)\| \le 4^3\|C(x)\|$, and similarly $\|C(z)^{-1}\| \le 4^3\|C(x)^{-1}\|$.

Now we establish Hölder continuity of C on Y_U. We fix $x, z \in Y_U$ and consider y and y' as above. Then by (5.7) we have

$$C(y) \circ C(x)^{-1} = (\mathrm{Id} + R_{x,y}^{\mathcal{A},s}) \circ C(x) \circ (\mathrm{Id} + R_{x,y}^{\mathcal{B},s}) \circ C(x)^{-1} =$$
$$= \mathrm{Id} + R_{x,y}^{\mathcal{A},s} + C(x) \circ R_{x,y}^{\mathcal{B},s} \circ C(x)^{-1} + R_{x,y}^{\mathcal{A},s} \circ C(x) \circ R_{x,y}^{\mathcal{B},s} \circ C(x)^{-1}.$$

Since C and C^{-1} are bounded on Y_U, it follows that

$$\|C(y) \circ C(x)^{-1} - \mathrm{Id}\| \le c_1 \operatorname{dist}(x,y)^{\beta}.$$

Hence we have

$$d(C(x), C(y)) = \|C(x) - C(y)\| + \|C(x)^{-1} - C(y)^{-1}\| \le$$
$$\le \|C(x)C(y)^{-1} - \mathrm{Id}\| \cdot \|C(y)\| + \|C(x)^{-1}\| \cdot \|\mathrm{Id} - C(x)C(y)^{-1}\| \le$$
$$\le c_2 \operatorname{dist}(x,y)^{\beta}, \quad \text{where } c_2 \text{ does not depend on } x \text{ and } y.$$

Using similar estimates for y, y' and y', z and the local product structure of the stable and unstable manifolds we conclude that for all $x, z \in Y_U$,

$$d(C(x), C(z)) \le c_3 \operatorname{dist}(x,z)^{\beta}.$$

Thus we obtain Hölder continuity of C on a set of full measure $Y_1 \subseteq Y$. Let $Y_2 = \bigcap_{n=-\infty}^{\infty} f^n(Y_1)$. Then Y_2 is f-invariant and $\mathcal{A}(x) = C(fx) \circ \mathcal{B}(x) \circ C(x)^{-1}$ for all $x \in Y_2$. Since μ has full support and $\mu(Y_2) = 1$, the set Y_2 is dense in X. Hence C extends from Y_2 to a Hölder continuous conjugacy $\tilde{C}$ on X. $\qquad\square$

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