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COHOMOLOGY OF $GL(2,\mathbb{R})$-VALUED COCYCLES OVER HYPERBOLIC SYSTEMS

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Abstract. We consider Hölder continuous $GL(2,\mathbb{R})$-valued cocycles over a transitive Anosov diffeomorphism. We give a complete classification up to Hölder cohomology of cocycles with one Lyapunov exponent and of cocycles that preserve two transverse Hölder continuous sub-bundles. We prove that a measurable cohomology between two such cocycles is Hölder continuous. We also show that conjugacy of periodic data for two such cocycles does not always imply cohomology, but a slightly stronger assumption does. We describe examples that indicate that our main results do not extend to general $GL(2,\mathbb{R})$-valued cocycles.

1. Introduction.

In this paper we study cohomology of $GL(2,\mathbb{R})$-valued cocycles over a transitive Anosov diffeomorphism f of a compact manifold $\mathcal{M}$. Let A be Hölder continuous function from $\mathcal{M}$ to a metric group G. The map $\mathcal{A}:\mathcal{M}\times\mathbb{Z}\to G$ defined by

$$\mathcal{A}(x,0)=e_G,\quad \mathcal{A}(x,n)=A(f^{n-1}x)\cdots A(x),\ \text{ and }\ \mathcal{A}(x,-n)=\mathcal{A}(f^{-n}x,n)^{-1}$$

is called a G-valued *cocycle* over the $\mathbb{Z}$-action generated by f. The function $A(x)=\mathcal{A}(x,1)$ is called the *generator* of $\mathcal{A}$, and we will often refer to A as a cocycle.

Cocycles appear naturally in dynamical systems, and an important example is given by the derivative cocycle. If the tangent bundle of $\mathcal{M}$ is trivial, i.e. $T\mathcal{M}=\mathcal{M}\times\mathbb{R}^m$, then the differential df can be viewed as a cocycle

$$\mathcal{A}(x,n)=df_x^n\in GL(m,\mathbb{R})\quad\text{and}\quad A(x)=df_x.$$

More generally, one can consider the restriction of df to a Hölder continuous invariant sub-bundle of $T\mathcal{M}$, such as the stable or unstable sub-bundle. Hölder regularity of the cocycles is natural in this context, and it is also necessary to develop a meaningful theory, even in the case of $G=\mathbb{R}$.

Definition 1.1. Cocycles $\mathcal{A}$ and $\mathcal{B}$ are (measurably, continuously) *cohomologous* if there exists a (measurable, continuous) function $C:\mathcal{M}\to G$ such that

$$\mathcal{A}(x,n)=C(f^nx)\mathcal{B}(x,n)C(x)^{-1}\quad\text{ for all } n\in\mathbb{Z}\text{ and } x\in\mathcal{M},\tag{1}$$

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equivalently, for the generators A and B of $\mathcal{A}$ and $\mathcal{B}$ respectively,

$$A(x) = C(fx)B(x)C(x)^{-1} \quad \text{for all } x \in \mathcal{M}.$$

We refer to C as a *conjugacy* between $\mathcal{A}$ and $\mathcal{B}$. It is also called a transfer map.

Cocycles over hyperbolic systems and their cohomology have been extensively studied starting with the seminal work of A. Livšic [7, 8], and the research has been focused on the following questions.

Question 1. *Is every measurable solution C of (1) continuous?*

Measurability should be understood with respect to a suitable measure, for example the measure of maximal entropy or the invariant volume. Further, one can ask whether a continuous solution is smooth if the system and the cocycles are.

Clearly, continuous cohomology of two cocycles implies conjugacy of their periodic data. So it is natural to ask whether the converse it true.

Question 2. *Suppose that whenever $p = f^n p$, $\mathcal{A}(p,n) = C(p)\mathcal{B}(p,n)C^{-1}(p)$ for some $C(p) \in G$. Does it follow that $\mathcal{A}$ and $\mathcal{B}$ are continuously cohomologous?*

Without any assumptions on continuity of $C(p)$ the answer is negative in general. If $C(p)$ is Hölder continuous, conjugating $\mathcal{B}$ by C reduces the question to the following.

Question 3. *Suppose that $\mathcal{A}(p,n) = \mathcal{B}(p,n)$ whenever $f^n p = p$. Does it follow that $\mathcal{A}$ and $\mathcal{B}$ are continuously cohomologous?*

If G is $\mathbb{R}$ or an abelian group, positive answers to all these questions where given in [7, 8]. For abelian groups, Questions 2 and 3 are equivalent, moreover, the analysis reduces to the case when $\mathcal{B}$ is the identity cocycle. Even for non-abelian G, the case of $B = e_G$ has been studied most and by now is relatively well understood. In Questions 2 and 3, the assumptions become $\mathcal{A}(p,n) = e_G$, and for a Lie group G these questions where answered positively by B. Kalinin in [3]. Question 1 remains open in full generality, but it has been answered positively under additional assumptions. For example, M. Pollicott and C. P. Walkden in [12] assumed certain pinching of the cocycle, and M. Nicol and M. Pollicott in [10] assumed boundedness of the conjugacy.

For non-abelian G the question of cohomology of two arbitrary cocycles is much more difficult. Positive answers to Questions 1 and 3 were given by W. Parry [11] for compact G and, somewhat more generally, by K. Schmidt [13] when both cocycles have "bounded distortion". The non-compact case remains largely unexplored. The only results so far have been negative. Cocycles which are measurably but not continuously cohomologous were constructed in [12], and an example of cocycles with conjugate periodic data that are not continuously cohomologous was given by M. Guysinsky in [2]. In these examples both cocycles can be made arbitrarily close to the identity, so no pinching can ensure positive results.

In this paper we go beyond the case of compact groups and consider $G = GL(2,\mathbb{R})$. We obtain positive results for two classes of cocycles. The first one consists of cocycles which have only one Lyapunov exponent for each ergodic f-invariant measure. Such cocycles can be identified by the periodic data: *for every periodic point $p = f^n p$, the eigenvalues of the matrix $\mathcal{A}(p,n)$ are equal in modulus* [4]. We give a complete classification of these cocycles up to Hölder cohomology, which shows that they can be viewed as either elliptic or parabolic. At the opposite end of the spectrum is the class of cocycles which preserve a pair of Hölder continuous transverse sub-bundles. It includes uniformly hyperbolic cocycles and, more generally, cocycles with dominated splitting. These cocycles are Hölder cohomologous to diagonal ones. Using the classification we obtain positive answers to Questions 1 and 3 and give a complete analysis of Question 2. In particular, we give an example of parabolic cocycles with $C(p)$ uniformly bounded that are not even measurably cohomologous.

The cocycles outside of these two classes can be viewed as non-uniformly hyperbolic. We revisit examples from [2, 12] that give negative answers to Questions 1 and 2 and indicate that continuous classification of such cocycles is unlikely. Question 3 for general $GL(2,\mathbb{R})$-valued cocycles remains open.

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2. **Statement of results.**

2.1. **Assumptions.** *In this paper, $\mathcal{M}$ is a compact connected Riemannian manifold, $f : \mathcal{M} \to \mathcal{M}$ is a transitive C^2 Anosov diffeomorphism, and $\mathcal{E} = \mathcal{M} \times \mathbb{R}^2$ is a trivial vector bundle with two-dimensional fibers. Sub-bundles of $\mathcal{E}$ are understood to be one-dimensional. We consider orientation-preserving Hölder continuous cocycles $A : \mathcal{M} \to GL(2,\mathbb{R})$ over f.*

Measurability is understood with respect to a mixing f-invariant probability measure on $\mathcal{M}$ with full support and local product structure, for example the measure of maximal entropy. Measurable objects are assumed to be defined almost everywhere with respect to such a measure. When we say that a measurable object is continuous, we mean that it coincides almost everywhere with a continuous one.

First we consider cocycles satisfying the following condition, which is equivalent to having only one Lyapunov exponent for each ergodic f-invariant measure [4].

Condition 2.1. For each periodic point $p = f^n p$ in $\mathcal{M}$, the eigenvalues of the matrix $\mathcal{A}(p,n) = A(f^{n-1}p) \cdots A(fp)A(p)$ are equal in modulus.

The following theorem gives a complete classification of these cocycles up to Hölder cohomology. It shows that they can be viewed as elliptic or parabolic. Orientation-preserving cocycles can have non-orientable invariant sub-bundles, as demonstrated by Example 8.1. Such a sub-bundle can be made orientable by passing to a double cover. For a double cover $P : \tilde{\mathcal{M}} \to \mathcal{M}$, the lift of A to $\tilde{\mathcal{M}}$ defined by $\tilde{A}(y) = A(P(y))$.

Theorem 2.2. *Any cocycle A satisfying Condition 2.1 belongs to exactly one of the five types below. Cocycles of different types are not Hölder continuously cohomologous.*

I. If A preserves exactly one Hölder continuous sub-bundle, which is orientable, then A is Hölder continuously cohomologous to a cocycle

$$A'(x) = k(x) \begin{bmatrix} 1 & \alpha(x) \\ 0 & 1 \end{bmatrix}, \text{ where } k(x) \neq 0 \text{ and } \alpha \text{ is not cohomologous to 0.}$$

I′. If A preserves exactly one Hölder continuous sub-bundle, which is not orientable, then there exists a cocycle A' as in I such that the lifts of A and A' to a double cover are Hölder continuously cohomologous.

II. If A preserves more than one orientable Hölder continuous sub-bundle, then A is Hölder continuously cohomologous to $A'(x) = k(x) \cdot Id$, where $k(x) \neq 0$.

II′. If A preserves more than one non-orientable Hölder continuous sub-bundle, then there exists a cocycle A' as in II such that the lifts of A and A' to a double cover are Hölder continuously cohomologous.

III. If A does not preserve any Hölder continuous sub-bundles then A is Hölder continuously cohomologous to

$$A'(x) = k(x) \begin{bmatrix} \cos\alpha(x) & -\sin\alpha(x) \\ \sin\alpha(x) & \cos\alpha(x) \end{bmatrix} \stackrel{def}{=} k(x)\, R(\alpha(x)), \quad \textit{where } k(x) > 0$$

and $\alpha : \mathcal{M} \to \mathbb{R}/2\pi\mathbb{Z}$ is such that $\alpha \bmod \pi$ is not cohomologous to 0 in $\mathbb{R}/\pi\mathbb{Z}$.

We refer to cocycles A' as *models*. In Section 5 we describe cohomology in $GL(2,\mathbb{R})$ for each type of the model cocycles giving explicit necessary and sufficient conditions. In particular, we show that measurable cohomology between the models is Hölder. These results together with Theorem 2.2 allow us to establish the following.

Theorem 2.3. *Suppose that cocycles A and B satisfy Condition 2.1. Then*

(i) Any measurable conjugacy between A and B is Hölder continuous;
(ii) If the diffeomorphism f and the cocycles A and B are C^k then a Hölder continuous conjugacy between A and B is C^r, where $r = k - \epsilon$ for $k \in \mathbb{N} \setminus \{1\}$ and any $\epsilon > 0$, and $r = k$ for $k = 1, \infty, \omega$.

Remark 1. Theorem 2.3 implies that the Hölder classification in Theorem 2.2 coincides with the measurable one.

Now we consider the question whether conjugacy of the periodic data for two cocycles implies cohomology.

Condition 2.4. A and B have conjugate periodic data, i.e. for every periodic point $p = f^n p$ in $\mathcal{M}$ there exists $C(p) \in GL(2,\mathbb{R})$ such that $\mathcal{A}(p,n) = C(p)\mathcal{B}(p,n)C^{-1}(p)$.

Proposition 1. *Let A and B be two cocycles of type II (or III) as in Theorem 2.2. If A and B satisfy Condition 2.4, then they are Hölder continuously cohomologous.*

Example 2.5 shows that, in general, Condition 2.4 does not imply measurable cohomology even when $C(p)$ is uniformly bounded.

Example 2.5. There exist cocycles $A(x) = \begin{bmatrix} 1 & \alpha(x) \\ 0 & 1 \end{bmatrix}$ and $B(x) = \begin{bmatrix} 1 & \beta(x) \\ 0 & 1 \end{bmatrix}$ arbitrarily close to the identity that satisfy Condition 2.4 with $C(p)$ uniformly bounded, but are not measurably cohomologous.

However, continuity of the conjugacy at a single point ensures Hölder cohomology of the cocycles. Continuity of C at p_0 can be replaced by a slightly weaker assumption that $\lim_{p\to z} C(p)$ exists at a point $z \in \mathcal{M}$.

Theorem 2.6. *Let A and B be two cocycles satisfying Condition 2.1. If A and B satisfy Condition 2.4 and $C(p)$ is continuous at a point p_0, then A and B are Hölder continuously cohomologous.*

Corollary 1. *Suppose that cocycles A and B satisfying Condition 2.1 have the same periodic data, i.e. $\mathcal{A}(p,n) = \mathcal{B}(p,n)$ whenever $f^n p = p$. Then A and B are Hölder continuously cohomologous.*

Next we consider cocycles that preserve two Hölder continuous transverse sub-bundles. These include uniformly hyperbolic cocycles, and more generally cocycles with dominated splitting. Such cocycles cannot be easily characterized by the periodic data. The only positive result is due to M. Guysinsky [2]. We recall that for a periodic point $p = f^n p$, the Lyapunov exponents of a cocycle A at p are given by

$$\lambda_p = n^{-1} \ln |\lambda'_p| \quad \text{and} \quad \mu_p = n^{-1} \ln |\mu'_p|,$$

where λ'_p and μ'_p are the eigenvalues of the matrix $\mathcal{A}(p,n)$.

Theorem 2.7 ([2])**.** *Let $A : \mathcal{M} \to GL(2,\mathbb{R})$ be a Hölder continuous cocycle over f. Suppose that there exist numbers $\lambda < \mu$ and a sufficiently small $\epsilon > 0$ such that*

$$|\lambda - \lambda_p| < \epsilon \quad \textit{and} \quad |\mu - \mu_p| < \epsilon \quad \textit{for every periodic point } p.$$

Then A preserves two transverse Hölder continuous sub-bundles. The smallness of ϵ depends only on the map f, the numbers λ and μ, and the Hölder exponent of A.

The assumptions of the theorem are quite strong, however, it is not sufficient just to have λ_p and μ_p contained in two disjoint closed intervals, as was demonstrated by A. Gogolev in [1].

Theorem 2.8. *Let A and B be two cocycles such that each one preserves two Hölder continuous transverse sub-bundles. Then*

(i) If the A-invariant sub-bundles are orientable, then A is Hölder cohomologous to a diagonal cocycle. If the sub-bundles are non-orientable, then there exists a diagonal cocycle A' such that the lifts of A and A' to a double cover are Hölder cohomologous.

(ii) Any measurable conjugacy between A and B is Hölder continuous.

(iii) If A and B have conjugate periodic data, then they are Hölder cohomologous.

Our main results do not extend to general $GL(2,\mathbb{R})$-valued cocycles, as demonstrated by the examples below, based on [2, 12]. These cocycles can be viewed as non-uniformly hyperbolic, they have two different exponents at (almost) all periodic points, however the exponents can be arbitrarily close to each other. The examples also indicate that a meaningful continuous classification of these cocycles is unlikely due to possibility of measurable but not continuous invariant sub-bundles.

Examples 2.9. Arbitrarily close to the identity, there exist smooth cocycles

$A(x) = \begin{bmatrix} \alpha(x) & \beta \\ 0 & 1 \end{bmatrix}$ and $B(x) = \begin{bmatrix} \alpha(x) & 0 \\ 0 & 1 \end{bmatrix}$ such that

(i) A and B are measurably, but not continuously cohomologous;

(ii) A preserves a measurable sub-bundle that is not Hölder continuous;

(iii) A and B have conjugate periodic data, but are not continuously cohomologous.

3. Preliminaries.

In this section we briefly introduce the main notions and results used in this paper.

3.1. Anosov diffeomorphisms.

Let f be a diffeomorphism of a compact connected Riemannian manifold $\mathcal{M}$. It is called *Anosov* if there exist a decomposition of the tangent bundle $T\mathcal{M}$ into two invariant continuous sub-bundles E^s and E^u, and constants $K > 0$, $\kappa > 0$ such that for all $n \in \mathbb{N}$, $\mathbf{v} \in E^s$, and $\mathbf{w} \in E^u$,

$$\|df^n(\mathbf{v})\| \le K e^{-\kappa n} \|\mathbf{v}\| \quad \text{and} \quad \|df^{-n}(\mathbf{w})\| \le K e^{-\kappa n} \|\mathbf{w}\|.$$

A diffeomorphism f is called *transitive* if there exists a point in $\mathcal{M}$ with dense orbit.

The simplest examples are given by *Anosov automorphisms* of tori. For a hyperbolic matrix F in $SL(n,\mathbb{Z})$, the map $F:\mathbb{R}^n\to\mathbb{R}^n$ projects to an automorphism f of the torus $\mathbb{T}^n=\mathbb{R}^n/\mathbb{Z}^n$, and f is clearly Anosov.

In the rest of this section, we assume that f *is a transitive Anosov diffeomorphism.* Abundance of periodic orbits is a key feature of such maps, and one of its strongest manifestations is the *Specification Property* [5, Theorem 18.3.9]:

Theorem 3.1. *For any $\epsilon>0$ there exists a positive integer M_ϵ such that given any collection of orbit segments*

$$O(x_l,n_l)=\{x_l,fx_l,\dots,f^{n_l-1}x_l\},\quad l=1,\dots,m,$$

there exists a periodic point p that ϵ-shadows each of the orbit segments with M_ϵ iterates between consecutive ones, more precisely, $f^{n_1+\cdots+n_m+mM_\epsilon}p=p$,

$$\begin{aligned}&\mathit{dist}(f^ip,f^ix_1)\le\epsilon,\;\; i=0,\dots,n_1-1,\quad \mathit{and}\\ &\mathit{for}\;\; l=2,\dots,m,\quad \mathit{dist}(f^{n_1+\cdots+n_{l-1}+(l-1)M_\epsilon+i}p,f^ix_l)\le\epsilon,\;\; i=0,\dots,n_l-1.\end{aligned}$$

We will use the following estimate for sums of a Hölder function along close orbits. This estimate is well-known, see for example [5, Proof of Lemma 19.2.2], and follows easily from exponential closeness of the orbit segments.

For a function $\alpha:\mathcal{M}\to\mathbb{R}$, we denote

$$\begin{aligned}\alpha^+(x,n)&=\alpha(x)+\alpha(fx)+\cdots+\alpha(f^{n-1}x),\\ \alpha^\times(x,n)&=\alpha(x)\alpha(fx)\cdots\alpha(f^{n-1}x).\end{aligned}\tag{2}$$

Lemma 3.2. *Let $\alpha:\mathcal{M}\to\mathbb{R}$ be a Hölder function with Hölder exponent σ. Then for any sufficiently small $\epsilon>0$ there exists a constant γ independent of n such that*

$$\mathit{if}\;\; \mathit{dist}(f^ix,f^iy)\le\epsilon,\;\; i=0,\dots,n-1,\;\; \mathit{then}\quad \left|\alpha^+(x,n)-\alpha^+(y,n)\right|\le\gamma\epsilon^\sigma.$$

Corollary 2. *Let $\beta:\mathcal{M}\to\mathbb{R}\setminus\{0\}$ be a Hölder function with Hölder exponent σ. Then for x and y as in Lemma 3.2 we have*

$$e^{-\gamma\epsilon^\sigma}\le\beta^\times(x,n)\cdot\big(\beta^\times(y,n)\big)^{-1}\le e^{\gamma\epsilon^\sigma}.$$

3.2. **Livšic Theorems** [7, 8]**.** Let $\alpha:\mathcal{M}\to\mathbb{R}$ be a Hölder function.

Theorem 3.3. *If $\alpha^+(p,n)=0$ whenever $f^np=p$, then there exists a Hölder function φ such that $\alpha(x)=\varphi(fx)-\varphi(x)$. Moreover, the conclusion still holds if $\alpha^+(p,n)=0$ for every periodic point p in a non-empty open f-invariant set.*

The stronger version was proved in [8, Section 5]. Alternatively, one can show using the Specification Property that the weaker assumption implies that $\alpha^+(p,n)=0$ for *all* periodic points.

Theorem 3.4. *Let μ be an ergodic probability measure on $\mathcal{M}$ with full support and local product structure. If φ is a μ-measurable function such that $\alpha(x)=\varphi(fx)-\varphi(x)$, then φ is Hölder, more precisely, φ coincides on a set of full measure with a Hölder function $\tilde\varphi$ such that $\alpha(x)=\tilde\varphi(fx)-\tilde\varphi(x)$ for all x.*

For a positive Hölder function β, Theorems 3.3 and 3.4 yield multiplicative counterparts: if $\beta^\times(p,n)=1$ whenever $f^np=p$, then $\beta(x)=\varphi(fx)/\varphi(x)$ for a Hölder function φ; and a measurable solution φ of the equation $\beta(x)=\varphi(fx)/\varphi(x)$ is Hölder.

3.3. **Conformal structures and conformal matrices.** A conformal structure $\mathcal{S}$ on $\mathbb{R}^2$ is a class of proportional inner products $\{\langle \mathbf{u}, \mathbf{v}\rangle_{\mathcal{S}}\}$. It can be identified with a real symmetric positive definite matrix S with determinant 1 via

$$\langle \mathbf{u}, \mathbf{v}\rangle_{\mathcal{S}} = \langle S\mathbf{u}, \mathbf{v}\rangle, \quad \text{where } \langle \cdot, \cdot \rangle \text{ is the standard inner product.}$$

The *standard* conformal structure is given by $\langle \cdot, \cdot \rangle$. The structure $\mathcal{S}$ can also be viewed as a class of proportional ellipses $\{E_{\mathcal{S}}\}$ given by the vectors of the same length with respect to $\langle \mathbf{u}, \mathbf{v}\rangle_{\mathcal{S}}$. For an invertible linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$, we denote by $A[\mathcal{S}]$ the conformal structure corresponding to the class of ellipses $\{AE_{\mathcal{S}}\}$, i.e. the matrix of $A[\mathcal{S}]$ is $\det(A^*A) \cdot (A^{-1})^* S (A^{-1})$.

Suppose that for each x in $\mathcal{M}$ we have a conformal structure $\mathcal{S}(x)$. This defines a conformal structure $\mathcal{S}$ on $\mathcal{M} \times \mathbb{R}^2$. Let $A : \mathcal{M} \to GL(2, \mathbb{R})$ be a cocycle. We say that $\mathcal{S}$ is A-invariant if $A(x)[\mathcal{S}(x)] = \mathcal{S}(fx)$ for all x.

A matrix is called *conformal* if it preserves the standard conformal structure, i.e it is a non-zero scalar multiple of an orthogonal matrix.

4. **Proof of Theorem 2.2.** The following statement serves as a motivation and plays an important role in our proofs. It is an immediate corollary of Propositions 2.1, 2.3, 2.6 and 2.7 in [4].

Proposition 2. *Suppose that a cocycle A satisfies Condition 2.1. Then*

(i) Any measurable A-invariant sub-bundle of $\mathcal{E}$ is Hölder continuous;
(ii) Any A-invariant measurable conformal structure on $\mathcal{E}$ is Hölder continuous;
(iii) The cocycle A preserves either a Hölder continuous sub-bundle of $\mathcal{E}$ or a Hölder continuous conformal structure on $\mathcal{E}$.

The following lemma shows that the number of invariant sub-bundles is an invariant of cohomology of the corresponding regularity. Since a continuous conjugacy also preserves orientability of invariant sub-bundles, it follows that cocycles of different types are not Hölder cohomologous.

Lemma 4.1. *Suppose that $A_2(x) = C(fx)A_1(x)C(x)^{-1}$ for two cocycles A_1 and A_2 and a measurable (Hölder) function $C : \mathcal{M} \to GL(2, \mathbb{R})$.*

(i) If A_1 preserves a measurable (Hölder) sub-bundle $\mathcal{V}_1$, then A_2 preserves a measurable (Hölder) sub-bundle $\mathcal{V}_2 = C\mathcal{V}_1$.
(ii) If A_1 preserves a measurable (Hölder) conformal structure $\mathcal{S}_1$, then A_2 preserves a measurable (Hölder) conformal structure $\mathcal{S}_2 = C[\mathcal{S}_1]$.

I-II′. Suppose that A preserves a non-orientable Hölder continuous sub-bundle $\mathcal{V}$.

Lemma 4.2. *There exists a double cover $\tilde{f} : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}$ of f such that the lift $\tilde{A}$ of A preserves an orientable sub-bundle $\tilde{\mathcal{V}}$ that projects to $\mathcal{V}$.*

Proof. We denote by $\pi_1(\mathcal{M})$ the fundamental group of $\mathcal{M}$ and consider a homomorphism $\rho : \pi_1(\mathcal{M}) \to \mathbb{Z}/2\mathbb{Z} = \{1, -1\}$ defined as follows: $\rho(\gamma) = 1$ if $\mathcal{V}$ is orientable along γ and -1 otherwise. Then $\ker \rho$ is a normal subgroup of index 2 in $\pi_1(\mathcal{M})$, and there exists a double cover $P : \tilde{\mathcal{M}} \to \mathcal{M}$ such that $P_*(\pi_1(\tilde{\mathcal{M}})) = \ker \rho$. The double cover has the property that the lift of a loop γ in $\mathcal{M}$ is also a loop in $\tilde{\mathcal{M}}$ if and only if $\mathcal{V}$ is orientable along γ. This property of γ is preserved by f. Indeed, the extension $(x, v) \mapsto (fx, A(x)v)$ gives a homeomorphism between the restrictions of $\mathcal{E}$ to γ and $f \circ \gamma$ which maps $\mathcal{V}$ to $\mathcal{V}$ and hence preserves orientability. It follows that f lifts to $\tilde{f} : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}$. The lift $\tilde{\mathcal{V}}(y) = \mathcal{V}(Py)$ of $\mathcal{V}$ to $\tilde{\mathcal{E}} = \tilde{\mathcal{M}} \times \mathbb{R}^2$ is orientable and is invariant under the lift $\tilde{A}(y) = A(Py)$. □

We choose a Hölder continuous unit vector fields $\mathbf{v}_1$ in $\tilde{\mathcal{V}}$ and $\bar{\mathbf{v}}_2$ orthogonal to $\mathbf{v}_1$. Then for the change of basis matrix $\bar{C}(y)$ from the standard basis to $\{\mathbf{v}_1(y), \bar{\mathbf{v}}_2(y)\}$,

$$\tilde{B}(y) \stackrel{\text{def}}{=} \bar{C}(\tilde{f}y)\, \tilde{A}(y)\, \bar{C}^{-1}(y) \quad \text{is upper triangular.} \tag{3}$$

Suppose that $P(y_1) = P(y_2)$ for $y_1, y_2 \in \tilde{\mathcal{M}}$. It follows from the construction of the double cover that $\mathbf{v}_1(y_1) = -\mathbf{v}_1(y_2)$, hence $\bar{\mathbf{v}}_2(y_1) = -\bar{\mathbf{v}}_2(y_2)$ and $\bar{C}(y_1) = -\bar{C}(y_2)$. Thus $\tilde{B}(y_1) = \tilde{B}(y_2)$, and $\tilde{B}$ is the lift of a cocycle B on $\mathcal{M}$ of the form

$$B(x) = k(x) \begin{bmatrix} 1 & * \\ 0 & g(x) \end{bmatrix}, \quad \text{where } \ k(x) \neq 0 \ \text{ and } \ g(x) > 0. \tag{4}$$

The lifts $\tilde{A}$ of A and $\tilde{B}$ of B are Hölder cohomologous via $\bar{C}$. The conjugacy $\bar{C}$ does not project to $\mathcal{M}$ in $GL(2,\mathbb{R})$, but does in $GL(2,\mathbb{R})/\{\pm\text{Id}\}$. It follows that A and B have the same number of invariant sub-bundles. Also, for any periodic point $p = f^n p$ the eigenvalues $k^\times(p,n)$ and $k^\times(p,n)g^\times(p,n)$ of the matrix $\mathcal{B}(p,n)$ have the same modulus, and hence $g^\times(p,n) = 1$. By Theorem 3.3, there exists a Hölder continuous function φ such that $g(x) = \varphi(fx)/\varphi(x)$. Rescaling the second coordinate by a factor $1/\varphi(x)$, we obtain a cocycle $A'(x)$ cohomologous to $B(x)$ of the form

$$A'(x) \;=\; k(x) \begin{bmatrix} 1 & \alpha(x) \\ 0 & 1 \end{bmatrix}, \quad \text{where } \ k(x) \neq 0. \tag{5}$$

The lifts $\tilde{A}$ of A and $\tilde{A}'$ of A' to $\tilde{\mathcal{M}}$ are cohomologous via a Hölder continuous function $\tilde{C} : \tilde{\mathcal{M}} \to GL(2,\mathbb{R})$. It is clear from the construction that

$$\text{if } \ P(y_1) = P(y_2) \ \text{ for } \ y_1 \neq y_2 \in \tilde{\mathcal{M}}, \ \text{ then } \ \tilde{C}(y_1) = -\tilde{C}(y_2). \tag{6}$$

If A has an orientable invariant sub-bundle, we obtain B as in (4) without passing to a double cover. In this case B, and hence A', are Hölder cohomologous to A.

The following lemma completes the analysis of the cases I-II$'$.

Lemma 4.3. *For A' as in (5) the following statements are equivalent*

(i) A' preserves at least two Hölder continuous sub-bundles;

(ii) α is Hölder cohomologous to 0;

(iii) A' is Hölder cohomologous to a scalar cocycle $k(x)Id$;

(iv) A' preserves infinitely many Hölder continuous sub-bundles.

Proof. Suppose that A' preserves two continuous sub-bundles. The set $\mathcal{N} \subset \mathcal{M}$ where the sub-bundles do not coincide is non-empty, open and f-invariant. Also, for every periodic point $p = f^n p$ in $\mathcal{N}$,

$$\mathcal{A}'(p,n) = k^\times(p,n) \begin{bmatrix} 1 & \alpha^+(p,n) \\ 0 & 1 \end{bmatrix} \ \text{ preserves two lines, and hence } \alpha^+(p,n) = 0.$$

By Theorem 3.3, α is Hölder cohomologous to 0, i.e. $\alpha(x) = s(fx) - s(x)$ for a Hölder function s. If such a function s exists, then

$$A'(x) = C(fx) \cdot l(x)\,\text{Id} \cdot C(x)^{-1}, \ \text{ where } C(x) = \begin{bmatrix} 1 & s(x) \\ 0 & 1 \end{bmatrix}, \ \text{ and (iv) follows.}$$

□

We note that A-invariant sub-bundles are either all orientable or all non-orientable. Indeed if a sub-bundle $\mathcal{V}$ is orientable, then we can obtain a continuous conjugacy between A and $k(x)$Id, which implies that all A-invariant sub-bundles are orientable.

This completes the classification of cocycles that have an invariant sub-bundle. For future reference in the proof of the Theorem 2.6 we make the following remark.

Remark 2. Let A be a cocycle of type I$'$ or II$'$ and let A' be its model. Then there is a periodic point $p = f^n p$ such that matrices $\mathcal{A}(p,n)$ and $\mathcal{A}'(p,n)$ are not conjugate.

Proof. Let $p = f^n p$ be a periodic point in $\mathcal{M}$ for which $\tilde{f}^n q_1 = q_2$, where q_1 and q_2 are the lifts of p. Then by (6) we have $C(\tilde{f}^n q_1) = \tilde{C}(q_2) = -\tilde{C}(q_1)$ and hence

$$\mathcal{A}'(p,n) = \tilde{\mathcal{A}}'(q_1,n) = \tilde{C}(\tilde{f}^n q_1)\,\tilde{\mathcal{A}}(q_1,n)\,\tilde{C}^{-1}(q_1) = -\tilde{C}(q_1)\,\mathcal{A}(p,n)\,\tilde{C}^{-1}(q_1).$$

If $\mathcal{A}(p,n)$ and $\mathcal{A}'(p,n)$ are conjugate, so are $\mathcal{A}(p,n)$ and $-\mathcal{A}(p,n)$, which is impossible.

Existence of such a point p can be easily obtained. The two lifts $\tilde{f}_1$ and $\tilde{f}_2$ of f to $\tilde{\mathcal{M}}$ satisfy $\tilde{f}_1 = i \circ \tilde{f}_2$, where i is the involution of the cover. Moreover, both lifts commute with i, and hence $\tilde{f}_1^n = i^n \circ \tilde{f}_2^n$. Hence for a periodic point of an odd period n, one of the lifts has the desired property. In fact, since both lifts have points of odd periods, such a point p exists for each lift. $\square$

III. Since A does not preserve any sub-bundles, by Proposition 2 (iii), A preserves a Hölder continuous conformal structure on $\mathcal{E}$. That is, for every x in $\mathcal{M}$, there is an inner product $\langle\cdot,\cdot\rangle_x$ such that

$$\langle A(x)\mathbf{u}, A(x)\mathbf{v}\rangle_{fx} = k_x\langle\mathbf{u},\mathbf{v}\rangle_x \quad \text{and} \quad \langle\mathbf{u},\mathbf{v}\rangle_x = \langle S(x)\mathbf{u},\mathbf{v}\rangle \ \text{ for all } \mathbf{u},\mathbf{v}\in\mathcal{E}_x,$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product and $S(x)$ is a real symmetric positive definite matrix that depends Hölder continuously on x. For such $S(x)$ there exists a unique symmetric positive definite matrix $C(x)$ satisfying $S(x) = C^2(x)$, which also depends Hölder continuously on x. Then $\langle\mathbf{u},\mathbf{v}\rangle_x = \langle S(x)\mathbf{u},\mathbf{v}\rangle = \langle C(x)\mathbf{u}, C(x)\mathbf{v}\rangle$ and hence

$$\langle C(fx)A(x)\mathbf{u}, C(fx)A(x)\mathbf{v}\rangle = \langle A(x)\mathbf{u}, A(x)\mathbf{v}\rangle_{fx} = k_x\langle\mathbf{u},\mathbf{v}\rangle_x = k_x\langle C(x)\mathbf{u}, C(x)\mathbf{v}\rangle.$$

Denoting $\mathbf{u}' = C(x)\mathbf{u}$ and $\mathbf{v}' = C(x)\mathbf{v}$, we obtain

$$\langle C(fx)A(x)C(x)^{-1}\mathbf{u}', C(fx)A(x)C(x)^{-1}\mathbf{v}'\rangle = k_x\langle\mathbf{u}',\mathbf{v}'\rangle \ \text{ for all } \mathbf{u}',\mathbf{v}'\in\mathcal{E}_x.$$

Thus $A'(x) = C(fx)A(x)C(x)^{-1}$ is Hölder continuous and conformal. Since $A'(x)$ is orientation-preserving, it is a scalar multiple of a rotation, i.e. $A'(x) = k(x)R(\alpha(x))$. Replacing α by $\alpha+\pi$ if necessary, we can assume that k is positive on $\mathcal{M}$.

It follows from the lemma below that, since A' does not preserve any invariant sub-bundles, the function $\alpha(x)\,(\mathrm{mod}\ \pi)$ is not cohomologous to 0.

Lemma 4.4. *Let $A'(x) = k(x)\,R(\alpha(x))$ and $k(x) \neq 0$. If A' preserves more than one measurable conformal structure, then $\alpha\,(\mathrm{mod}\ \pi)$ is cohomologous to 0 in $\mathbb{R}/\pi\mathbb{Z}$ and A' preserves infinitely many conformal structures and infinitely many sub-bundles.*

Proof. Suppose that A' preserves a measurable, and hence continuous, conformal structure $\mathcal{S}$ different from the standard one $\mathcal{S}_0$. The set $\mathcal{N}$ where $\mathcal{S} \neq \mathcal{S}_0$ is non-empty, open and invariant. At every periodic point $p = f^n p$ in $\mathcal{N}$, the matrix $\mathcal{A}'(p,n) = k^\times(p,n)\,R(\alpha^+(p,n))$ preserves a non-circular ellipse up to scaling, and hence $\mathcal{A}'(p,n) = \pm k\,\mathrm{Id}$. It follows that $\alpha^+(p,n) = 0\,(\mathrm{mod}\ \pi)$ for any periodic point in $\mathcal{N}$, and by Theorem 3.3, $\alpha\,(\mathrm{mod}\ \pi)$ is cohomologous to 0 in $\mathbb{R}/\pi\mathbb{Z}$.

Let $\bar{\alpha}(x) = \alpha(x)\ (\mathrm{mod}\ \pi)$ and let $\bar{A}' = k(x)\,R(\bar{\alpha}(x))$ be the projection of A' to $GL(2,\mathbb{R})/\{\pm\mathrm{Id}\}$. If $\bar{\alpha}$ is cohomologous to 0, i.e. $\bar{\alpha}(x) = \bar{s}(fx) - \bar{s}(x)$ for a Hölder continuous function $\bar{s}: \mathcal{M} \to \mathbb{R}/\pi\mathbb{Z}$, then in $GL(2,\mathbb{R})/\{\pm\mathrm{Id}\}$ we have

$$\bar{A}'(x) = \bar{C}(fx)\cdot k(x)\,\mathrm{Id}\cdot\bar{C}(x)^{-1}, \quad \text{where } \bar{C}(x) = R(\bar{s}(x)).$$

Hence $\bar{A}'$ and A' preserve infinitely many conformal structures and sub-bundles. $\square$

5. **Cohomology of the model cocycles.** First we consider cohomology of non-trivial upper triangular cocycles.

Proposition 3. *Let* $A(x) = k(x)\begin{bmatrix} 1 & \alpha(x) \\ 0 & 1 \end{bmatrix}$ *and* $B(x) = l(x)\begin{bmatrix} 1 & \beta(x) \\ 0 & 1 \end{bmatrix}$, *where* $k(x)$, $l(x) \neq 0$ *for all* x, *and* α, β *are not cohomologous to 0. Then*

(i) Any measurable conjugacy between A *and* B *is Hölder and upper triangular.*

(ii) A *and* B *are (measurably or Hölder) cohomologous if and only if there exist Hölder functions* $\varphi(x)$ *and* $s(x)$ *and a constant* $c \neq 0$ *such that* $k(x)/l(x) = \varphi(fx)/\varphi(x)$ *and* $\alpha(x) - c\beta(x) = s(fx) - s(x)$.

(iii) A measurable function $D(x)$ *satisfies* $A(x) = D(fx)A(x)D(x)^{-1}$ *if and only if* D *is a constant upper triangular matrix with equal diagonal entries.*

The last part of the proposition describes the *centralizer*, or the set of self-conjugacies of A. We discuss this set and its connections to conjugacies in Section 5.1.

Proof. **(i, ii)** Let C be a measurable function such that $A(x) = C(fx)B(x)C(x)^{-1}$. We can assume that the set where C is defined is f-invariant. Clearly, A preserves the sub-bundle $\mathcal{E}_1$ spanned by the first coordinate vector, and by Lemma 4.3, it is the only measurable invariant sub-bundle for A. Since $\mathcal{E}_1$ is B-invariant, $C(x)\mathcal{E}_1$ is a measurable A-invariant sub-bundle. If follows that $C(x)\mathcal{E}_1 = \mathcal{E}_1$ and hence the matrix $C(x)$ is upper triangular a.e. Thus for all x in an invariant set X of full measure, $C(x) = \varphi(x)\begin{bmatrix} r(x) & s(x) \\ 0 & 1 \end{bmatrix}$. Then $A(x) = C(fx)B(x)C(x)^{-1}$ yields

$$k(x)\begin{bmatrix} 1 & \alpha(x) \\ 0 & 1 \end{bmatrix} = \frac{l(x)\varphi(fx)}{\varphi(x)}\begin{bmatrix} \frac{r(fx)}{r(x)} & -\frac{r(fx)}{r(x)}s(x) + r(fx)\beta(x) + s(fx) \\ 0 & 1 \end{bmatrix}.$$

It follows that $k(x)/l(x) = \varphi(fx)/\varphi(x)$ on X. The functions k and l have constant sign on $\mathcal{M}$, moreover they are of the same sign. Otherwise, $\operatorname{sign}\varphi(fx) = -\operatorname{sign}\varphi(x)$ and hence for the sets $X_\pm = \{x \in X : \operatorname{sign}\varphi(x) = \pm 1\}$ we have $f(X_+) = X_-$, which contradicts mixing. It follows from Theorem 3.4 that the measurable function φ is Hölder and we have $k(x)/l(x) = \varphi(fx)/\varphi(x)$ for all x in $\mathcal{M}$.

Since $r(fx)/r(x) = 1$, the function r is invariant, and hence constant a.e. Then

$$-s(x) + c\beta(x) + s(fx) = \alpha(x), \quad \text{equivalently} \quad \alpha(x) - c\beta(x) = s(fx) - s(x) \text{ a.e.,}$$

and hence by Theorem 3.4 the measurable function $s(x)$ is Hölder.

Thus, if there is a measurable conjugacy between A and B then it is of the form

$$C(x) = \varphi(x)\begin{bmatrix} c & s(x) \\ 0 & 1 \end{bmatrix}, \tag{7}$$

where $\varphi(x)$ and $s(x)$ are Hölder continuous functions such that

$$k(x)/l(x) = \varphi(fx)/\varphi(x) \quad \text{and} \quad \alpha(x) - c\beta(x) = s(fx) - s(x) \quad \text{for all } x \text{ in } \mathcal{M}.$$

Conversely, if such c, φ, and s exist, then C is a Hölder conjugacy between A and B.

(iii) If a measurable function $D(x)$ satisfies $A(x) = D(fx)A(x)D(x)^{-1}$, then it is of the form (7), where $\varphi(fx)/\varphi(x) = 1$ and $(1-c)\alpha(x) = s(fx) - s(x)$. This implies that φ is constant and, since α is not cohomologous to 0, $c = 1$ and hence s

is constant. Thus $D(x) = D = d\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$, Conversely, any such matrix D satisfies the equation. □

The case of scalar cocycles is simple.

Proposition 4. *Let $A(x) = k(x)Id$ and $B(x) = l(x)Id$, where $k(x), l(x) \neq 0$. Then*

(i) Any measurable conjugacy between A and B is of the form $\varphi(x)C_0$, where $\varphi(x)$ is a Hölder continuous function such that $\varphi(fx)/\varphi(x) = k(x)/l(x)$.

(ii) A measurable function $D(x)$ satisfies $A(x) = D(fx)A(x)D(x)^{-1}$ if and only if D is constant.

Now we consider non-trivial conformal cocycles.

Proposition 5. *Let $A(x) = k(x)\,R(\alpha(x))$ and $B(x) = l(x)\,R(\beta(x))$, where $k(x)$, $l(x) > 0$ for all x, and $\alpha, \beta : \mathcal{M} \to \mathbb{R}/2\pi\mathbb{Z}$ are such that α and β (mod π) are not cohomologous to 0 in $\mathbb{R}/\pi\mathbb{Z}$. Then*

(i) Any measurable conjugacy between A and B is Hölder and conformal.

(ii) A and B are (measurably or Hölder) cohomologous if and only if there exist Hölder continuous functions $\varphi : \mathcal{M} \to \mathbb{R}$ and $s : \mathcal{M} \to \mathbb{R}/2\pi\mathbb{Z}$ and a constant $c = \pm 1$ such that $k(x)/l(x) = \varphi(fx)/\varphi(x)$ and $\alpha(x) - c\beta(x) = s(fx) - s(x)$.

(iii) A measurable function $D(x)$ satisfies $A(x) = D(fx)A(x)D(x)^{-1}$ if and only if $D(x) = D$ is a constant scalar multiple of a rotation.

It is clear from the proof that $c = 1$ and $c = -1$ in (ii) correspond to the conjugacy being orientation-preserving and orientation-reversing respectively.

Proof. **(i, ii)** Hölder continuity of a measurable conjugacy can be obtained as a corollary of the result by K. Schmidt [13] on cocycles of bounded distortion. However, we will obtain it independently as a part of our proof.

Let C be a measurable conjugacy between A and B. The cocycle B preserves the standard conformal structure $\mathcal{S}_0$, and hence $C[\mathcal{S}_0]$ is a measurable invariant conformal structure for A. By Lemma 4.4, $\mathcal{S}_0$ is the only such conformal structure. It follows that $C[\mathcal{S}_0] = \mathcal{S}_0$ and hence C is conformal a.e. Since the set where C is orientation-preserving is f-invariant, by ergodicity, C is either orientation-preserving a.e. or orientation-reversing a.e. If C is orientation-preserving, we can write $C(x) = \varphi(x)R(s(x))$, and the equation $A(x) = C(fx)B(x)C^{-1}(x)$ yields

$$k(x)\,R(\alpha(x)) = (l(x)\,\varphi(fx)/\varphi(x)) \cdot R(\beta(x) + s(fx) - s(x)).$$

It follows that for almost every x

$$k(x)/l(x) = \varphi(fx)/\varphi(x) \quad \text{and} \quad \alpha(x) - \beta(x) = s(fx) - s(x). \tag{8}$$

By Theorem 3.4, the measurable functions φ and s are Hölder continuous. Conversely, if such functions φ and s exist, then $C(x) = \varphi(x)R(s(x))$ is a Hölder conjugacy between A and B. If $C(x)$ is orientation-reversing, it is a scalar multiple of a reflection,

$$C(x) = \varphi(x)\begin{bmatrix} \cos s(x) & \sin s(x) \\ \sin s(x) & -\cos s(x) \end{bmatrix} \stackrel{\text{def}}{=} \varphi(x)\,Q(s(x)).$$

It follows that for almost every x

$$k(x)/l(x) = \varphi(fx)/\varphi(x) \quad \text{and} \quad \alpha(x) + \beta(x) = s(fx) - s(x), \tag{9}$$

and hence φ and s are Hölder continuous.

(iii) Let $D(x)$ be a measurable function satisfying $A(x) = D(fx)A(x)D(x)^{-1}$. If $D(x)$ is orientation-preserving, then $D(x) = \varphi(x)R(s(x))$, and by (8) we have $\varphi(fx)/\varphi(x) = 1$ and $s(fx) - s(x) = 0$. Hence $\varphi(x) = d$ and $s(x) = s$ are constant, and $D(x) = D = d\,R(s)$. Conversely, any such matrix D satisfies the equation.

If $D(x)$ is orientation-reversing, we obtain (9) with $k = l$ and $\alpha = \beta$. The latter implies that α is cohomologous to 0, which contradicts the assumption. □

5.1. Centralizers of cocycles and connection to conjugacies.

Let $A, B : \mathcal{M} \to G$ be two cocycles. The centralizer of A is the set

$$Z(A) = \{D : \mathcal{M} \to G \mid \; A(x) = D(fx)A(x)D(x)^{-1}\}.$$

It is easy to see that $Z(A)$ is a group with respect to pointwise multiplication. We denote by $\mathrm{Conj}(A, B)$ the set of conjugacies between A and B, i.e.

$$\mathrm{Conj}\,(A, B) = \{C \mid A(x) = C(fx)B(x)C^{-1}(x)\}.$$

Both sets can be considered in any regularity. The following properties can be verified by a direct computation.

(i) $\mathrm{Conj}\,(A, B) = Z(A) \cdot C$, where $C \in \mathrm{Conj}\,(A, B)$.
(ii) $Z(A) = C \cdot Z(B) \cdot C^{-1}$, where $C \in \mathrm{Conj}\,(A, B)$.

In Propositions 3, 4, and 5 we described the centralizers of model upper triangular, scalar, and conformal cocycles respectively. In each case, the centralizer is a subgroup of the constant matrix functions. It follows that a conjugacy between a model cocycle A and a measurable cocycle B is unique up to left multiplication by a constant matrix of the corresponding type. Property (ii) gives, in particular, a description of the centralizer of a cocycle that is cohomologous to a model one.

6. Proofs of Theorem 2.3, Proposition 1 and Theorem 2.6.

6.1. Proof of Theorem 2.3.

(i) Lemma 4.1 shows that the number of measurable invariant sub-bundles is an invariant of measurable cohomology. By Proposition 2, measurable sub-bundles are Hölder continuous, and it follows that cocycles with different number of Hölder continuous invariant sub-bundles cannot be measurably cohomologous. In the previous section we established Hölder continuity of a measurable conjugacy for the three types of model cocycles, and it remains to reduce the general case to the model one.

Let C be a measurable conjugacy between A and B. Suppose that each cocycle preserves exactly one sub-bundle, which is orientable. Then by Theorem 2.2, A and B are Hölder cohomologous to model triangular cocycles A' and B'. Thus we have

$$A' \overset{C_A}{\sim} A \overset{C}{\sim} B \overset{C_B}{\sim} B'.$$

By Proposition 3 (i), the measurable conjugacy $C_A C C_B$ between A' and B' is Hölder, and hence so is C.

Suppose that A and B preserve unique sub-bundles, $\mathcal{V}_A$ and $\mathcal{V}_B$ respectively, and at least one of the sub-bundles is not orientable. We pass to a double cover to make $\mathcal{V}_A$ orientable and then, if necessary, we pass to a double cover again to make the lift of $\mathcal{V}_B$ orientable. Thus we obtain lifts $\tilde{A}$ of A and $\tilde{B}$ of B that are measurably conjugate via a lift $\tilde{C}$ of C and preserve unique sub-bundles that are orientable. By the argument above, $\tilde{C}$ is Hölder continuous, and hence so is C.

The result for cocycles with at least two invariant sub-bundles, and with no invariant sub-bundles, is obtained similarly.

(ii) This follows from a result by V. Niţică and A. Török [9, Theorem 2.4]. Indeed, it is easy to see that for any model cocycle A we have

$$\lim_{n\to\infty} \sup_{x\in\mathcal{M}} \|\mathrm{Ad}_{\mathcal{A}(x,n)}\|^{1/n} = 1 \quad \text{and} \quad \lim_{n\to\infty} \inf_{x\in\mathcal{M}} \|\mathrm{Ad}_{\mathcal{A}(x,n)^{-1}}\|^{-1/n} = 1,$$

where Ad is the adjoint. It follows that the same holds for any cocycle satisfying Condition 2.1. Hence the theorem can be applied with $G = GL(2,\mathbb{R})$ and $\alpha_0 = 0$. □

6.2. **Proof of Proposition 1.**

First we consider two model conformal cocycles A and B as in Proposition 5 satisfying Condition 2.4. Since $A(x) = k(x)\,R(\alpha(x))$ and $B(x) = l(x)\,R(\beta(x))$,

$$\mathcal{A}(p,n) = k^{\times}(p,n)\cdot R(\alpha^{+}(p,n)) \quad \text{and} \quad \mathcal{B}(p,n) = l^{\times}(p,n)\cdot R(\beta^{+}(p,n)),$$

which implies that $k^{\times}(p,n) = l^{\times}(p,n)$.

Suppose that $\alpha^{+}(p,n) \neq 0 \,(\mathrm{mod}\ \pi)$. Then $\beta^{+}(p,n) \neq 0 \,(\mathrm{mod}\ \pi)$, and both $\mathcal{A}(p,n)$ and $\mathcal{B}(p,n)$ preserve only the standard conformal structure. Hence, depending on the sign of the determinant, $C(p)$ is either a rotation or a reflection. In Lemma 6.1 below we show that $\det C(p)$ has the same sign for all such p. In the case of a rotation,

$$\mathcal{A}(p,n) = R(s)\cdot\mathcal{B}(p,n)\cdot R(-s) = \mathcal{B}(p,n) \quad \text{and hence } \alpha^{+}(p,n) = \beta^{+}(p,n).$$

In the case of a reflection, $\mathcal{A}(p,n) = Q(s)\cdot\mathcal{B}(p,n)\cdot Q(s) = l^{\times}(p,n)\cdot R(-\beta^{+}(p,n))$, which implies that $\alpha^{+}(p,n) = -\beta^{+}(p,n)$.

If $\alpha^{+}(p,n) = 0\,(\mathrm{mod}\ \pi)$, then $\mathcal{B}(p,n) = \mathcal{A}(p,n)$, and hence $\alpha^{+}(p,n) = \beta^{+}(p,n) = 0$ or π. This implies that $\alpha^{+}(p,n) - \beta^{+}(p,n) = 0$ and $\alpha^{+}(p,n) + \beta^{+}(p,n) = 0$ in $\mathbb{R}/2\pi\mathbb{Z}$.

Thus there exists a constant $c = \pm 1$ such that $\alpha^{+}(p,n) - c\beta(p,n) = 0$ whenever $f^n p = p$. Hence $k(x)/l(x) = \varphi(fx)/\varphi(x)$ and $\alpha(x) - c\beta(x) = s(fx) - s(x)$ for Hölder functions, and A and B are Hölder cohomologous.

Lemma 6.1. *If $C(p)$ satisfies Condition 2.4 for A and B as Proposition 5, then $\det C(p)$ has the same sign for all p where $\alpha^{+}(p,n) \neq 0$ and $\beta^{+}(p,n) \neq 0$ (mod π).*

Proof. Suppose that there exist two such points $p_1 = f^{n_1}p_1$ and $p_2 = f^{n_2}p_2$ with $\det C(p_1) > 0$ and $\det C(p_2) < 0$. Then by the above argument,

$$\alpha^{+}(p_1,n_1) = \beta^{+}(p_1,n_1) \quad \text{and} \quad \alpha^{+}(p_2,n_2) = -\beta^{+}(p_2,n_2). \tag{10}$$

We use the Specification Property, Theorem 3.1. We consider two orbit segments

$$\{p_1, fp_1, \dots, f^{kn_1-1}p_1\} \quad \text{and} \quad \{p_2, fp_2, \dots, f^{kn_2-1}p_2\}. \tag{11}$$

Let $\epsilon > 0$. Then there exists M_ϵ independent of k and a periodic point q such that

$$\begin{aligned} &\mathrm{dist}\,(f^i q, f^i p_1) < \epsilon \quad \text{for } i = 0,\dots,kn_1-1,\\ &\mathrm{dist}\,(f^{kn_1+M_\epsilon+i}q, f^i p_2) < \epsilon \quad \text{for } i = 0,\dots,kn_2-1,\\ &f^n q = q, \quad \text{where } n = kn_1 + kn_2 + 2M_\epsilon. \end{aligned} \tag{12}$$

Let σ be a Hölder exponent of α and β, $m_\alpha = \max_{\mathcal{M}} |\alpha(x)|$, and $m_\beta = \max_{\mathcal{M}} |\beta(x)|$. Then it is easy to see using Lemma 3.2 that

$$\begin{aligned} &|\alpha^{+}(q,n) - k\cdot\alpha^{+}(p_1,n_1) - k\cdot\alpha^{+}(p_2,n_2)| \le \gamma_\alpha\epsilon^\sigma + 2M_\epsilon m_\alpha \quad \text{and}\\ &|\beta^{+}(q,n) - k\cdot\beta^{+}(p_1,n_1) - k\cdot\beta^{+}(p_2,n_2)| \le \gamma_\beta\epsilon^\sigma + 2M_\epsilon m_\beta, \end{aligned}$$

where constants γ_α and γ_β are independent of k. By (10) one can choose a sufficiently large k so that $\alpha^+(q,n) \neq \beta^+(q,n)$ and $\alpha^+(q,n) \neq -\beta^+(q,n)$, and hence $\mathcal{A}(q,n)$ and $\mathcal{B}(q,n)$ are not conjugate. ☐

This completes the proof for the case of two model conformal cocycles. Suppose that A and B are two cocycles of type III satisfying Condition 2.4. Then by Theorem 2.2, A and B are Hölder cohomologous to model conformal cocycles A' and B', and it is easy to see that A' and B' also satisfy Condition 2.4. Hence A' and B' are Hölder cohomologous, and so are A and B. The result for cocycles of type II follows similarly from Theorem 2.2 and Proposition 4. ☐

6.3. **Proof of Theorem 2.6.** In the rest of this section, we consider

Condition 6.2. For every periodic point $p = f^n p$ there exists $C(p) \in GL(2,\mathbb{R})$ such that $\mathcal{A}(p,n) = C(p)\mathcal{B}(p,n)C^{-1}(p)$, and $C(p)$ is continuous at a point p_0.

Proposition 6. *Let A and B be triangular cocycles as in Proposition 3. If A and B satisfy Condition 6.2, then A and B are Hölder cohomologous.*

Proof. For every periodic point $p = f^n p$ we have

$$\mathcal{A}(p,n) = k^\times(p,n) \begin{bmatrix} 1 & \alpha^+(p,n) \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathcal{B}(p,n) = l^\times(p,n) \begin{bmatrix} 1 & \beta^+(p,n) \\ 0 & 1 \end{bmatrix}.$$

Since the matrices are conjugate, $k^\times(p,n) = l^\times(p,n)$. Below we show that there exists a constant c such that $\alpha^+(p,n) = c\beta^+(p,n)$ whenever $f^n p = p$. By Theorem 3.3, $k(x)/l(x) = \varphi(fx)/\varphi(x)$ and $\alpha(x) - c\beta(x) = s(fx) - s(x)$ for Hölder functions φ and s, and hence A and B are Hölder cohomologous by Proposition 3.

Suppose that there exist points $p_1 = f^{n_1}p_1$ and $p_2 = f^{n_2}p_2$ such that

$$\alpha^+(p_{1,2}, n_{1,2}) \neq 0, \;\; \beta^+(p_{1,2}, n_{1,2}) \neq 0, \quad \frac{\alpha^+(p_1,n_1)}{\beta^+(p_1,n_1)} = c_1 \neq c_2 = \frac{\alpha^+(p_2,n_2)}{\beta^+(p_2,n_2)}. \tag{13}$$

Let $z \in \mathcal{M}$ and $\epsilon > 0$. We consider two orbit segments: $\{z\}$ and $\{p_1, fp_1, \dots f^{kn_1-1}p_1\}$. By the Specification Property, there exists a number M_ϵ independent of k and a point $q_1 = f^{t_1}q_1$, where $t_1 = kn_1 + 2M_\epsilon + 1$, such that

$$\text{dist}(q_1, z) < \epsilon \quad \text{and} \quad \text{dist}(f^{M_\epsilon+1+i}q_1,\, f^i p_1) < \epsilon \;\text{ for } i = 0, \dots, kn_1 - 1.$$

Let σ be a Hölder exponent of α and β, $m_\alpha = \max_{\mathcal{M}} |\alpha(x)|$ and $m_\beta = \max_{\mathcal{M}} |\beta(x)|$. It follows easily from Lemma 3.2 that there exists constants γ_α and γ_β independent of k such that

$$|\,\alpha^+(q_1,t_1) - k\cdot\alpha^+(p_1,n_1)\,| \le \gamma_\alpha\epsilon^\sigma + (2M_\epsilon+1)m_\alpha \quad \text{and}$$
$$|\,\beta^+(q_1,t_1) - k\cdot\beta^+(p_1,n_1)\,| \le \gamma_\beta\epsilon^\sigma + (2M_\epsilon+1)m_\beta.$$

Let $\delta = \frac{1}{3}|c_1 - c_2|$. Since $\alpha^+(p_1,n_1) \neq 0$ and $\beta^+(p_1,n_1) \neq 0$, by choosing a sufficiently large k, we can ensure that $\alpha^+(q_1,t_1) \neq 0$, $\beta^+(q_1,t_1) \neq 0$, and

$$\left| \frac{\alpha^+(q_1,t_1)}{\beta^+(q_1,t_1)} - c_1 \right| = \left| \frac{\alpha^+(q_1,t_1)}{\beta^+(q_1,t_1)} - \frac{k\cdot\alpha^+(p_1,n_1)}{k\cdot\beta^+(p_1,n_1)} \right| < \delta.$$

Similarly, using the orbit of p_2, in an ϵ-neighborhood of z we can find a periodic point q_2 of a period t_2 such that $\alpha^+(q_2,t_2) \neq 0$, $\beta^+(q_2,t_2) \neq 0$, and the ratio $\alpha^+(q_2,t_2)/\beta^+(q_2,t_2)$ is δ-close to c_2. It follows that at every point $z \in \mathcal{M}$, the ratio $\alpha^+(p,n)/\beta^+(p,n)$ has no limit, and in particular the ratio is discontinuous at every periodic point.

Suppose that $\alpha^+(p,n) \neq 0$ and $\beta^+(p,n) \neq 0$ and $\mathcal{A}(p,n) = C(p)\mathcal{B}(p,n)C(p)^{-1}$. Then $C(p)$ is upper triangular, and a direct calculation shows that it is of the form

$$C(p) = \varphi(p) \begin{bmatrix} \alpha^+(p,n)/\beta^+(p,n) & d(p) \\ 0 & 1 \end{bmatrix}.$$

Therefore, discontinuity of the ratio $\alpha^+(p,n)/\beta^+(p,n)$ implies discontinuity of C.

Thus no two periodic points satisfy (13). It follows that there exists a constant c such that $\alpha^+(p,n) = c\beta^+(p,n)$ at every periodic point, and hence A and B are Hölder cohomologous. □

Next we show that cocycles A and B of different types, as in Theorem 2.2, cannot satisfy Condition 6.2. Clearly, this is the case for cocycles with different number of invariant sub-bundles. The following lemma establishes this for cocycles with different orientability types of invariant sub-bundles.

Lemma 6.3. *Let A be a cocycle of type I (II) and B be a cocycle of type I′ (II′). Then A and B do not satisfy Condition 6.2.*

Proof. Suppose that cocycles A of type I and B of type I′ satisfy Condition 6.2. By Theorem 2.2, there exist model triangular cocycles A' and B' such that A' is Hölder cohomologous to A, and the lifts $\tilde{B}$ of B and $\tilde{B}'$ of B' to a double cover $\tilde{\mathcal{M}}$ are Hölder cohomologous. Clearly, the lifts $\tilde{A}$ of A and $\tilde{A}'$ of A' to $\tilde{\mathcal{M}}$ are also Hölder cohomologous. Since A and B satisfy Condition 6.2, so do $\tilde{A}$ and $\tilde{B}$ and hence the model cocycles $\tilde{A}'$ and $\tilde{B}'$. By Proposition 6, the cocycles $\tilde{A}'$ and $\tilde{B}'$ are Hölder cohomologous, and it follows from Lemma 6.4 below that so are A' and B'. Thus the cocycle B of type I′ and its model B' have conjugate periodic data, which contradicts Remark 2. A similar argument yields the result for cocycles of types II and II′. □

Lemma 6.4. *Let A and B be model triangular cocycles as in Proposition 3 and let $\tilde{A}$ and $\tilde{B}$ be their lifts to the same double cover. Then a Hölder conjugacy between $\tilde{A}$ and $\tilde{B}$ projects to a Hölder conjugacy between A and B.*

Proof. We denote the lifts of α and β by $\tilde{\alpha}$ and $\tilde{\beta}$. Since $\tilde{A}$ and $\tilde{B}$ are Hölder cohomologous, $\tilde{\alpha}^+(q,m) = c\tilde{\beta}^+(q,m)$ whenever $\tilde{f}^m q = q \in \tilde{\mathcal{M}}$. Let $p = f^n p \in \mathcal{M}$ and let $q \in \tilde{\mathcal{M}}$ be such that $p = P(q)$. Then $q = \tilde{f}^m(q)$ where m is either n or $2n$, and it follows that $\alpha^+(p,n) = c\beta^+(p,n)$. Similarly, $k^\times(p,n) = l^\times(p,n)$ whenever $f^n p = p$. Thus A and B are Hölder cohomologous. The discussion in Section 5.1 implies that a conjugacy between A and B, as well as between $\tilde{A}$ and $\tilde{B}$, is unique up to multiplication by a constant matrix. Hence a conjugacy between A and B is the projection of a conjugacy between $\tilde{A}$ and $\tilde{B}$. □

We conclude that if A and B satisfy Condition 6.2, then they are of the same type. By Proposition 1, it remains to consider cocycles of types I, I′, and II′. If A and B are of type I, they are Hölder cohomologous to model triangular cocycles A' and B', respectively. It follows that A' and B' also satisfy Condition 6.2. By Proposition 6, A' and B' are Hölder cohomologous, and hence so are A and B.

Let A and B be cocycles of type I′. It follows easily from Lemma 6.3 that their invariant sub-bundles can be made orientable by passing to the same double cover $\tilde{\mathcal{M}}$. The lifts $\tilde{A}$ and $\tilde{B}$ to $\tilde{\mathcal{M}}$ are of type I and satisfy Condition 6.2, and hence are Hölder cohomologous. By Theorem 2.2, there exist model triangular cocycles A'

and B' whose lifts to $\tilde{\mathcal{M}}$ are Hölder cohomologous to $\tilde{A}$ and $\tilde{B}$ respectively. Thus we have

$$\begin{array}{ccccccc} \tilde{A}' & \overset{\tilde{C}_A}{\sim} & \tilde{A} & \overset{\tilde{C}}{\sim} & \tilde{B} & \overset{\tilde{C}_B}{\sim} & \tilde{B}' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A' & & A & & B & & B' \end{array}$$

Let $y_1, y_2 \in \tilde{\mathcal{M}}$ be such that $P(y_1) = P(y_2)$. By (6), $\tilde{C}_A(y_1) = -\tilde{C}_A(y_2)$ and $\tilde{C}_B(y_1) = -\tilde{C}_B(y_2)$. By Lemma 6.4, the conjugacy $C' = \tilde{C}_A\tilde{C}\tilde{C}_B$ between $\tilde{A}'$ and $\tilde{B}'$ projects to $\mathcal{M}$, which means that $C'(y_1) = C'(y_2)$. Thus, $\tilde{C}(y_1) = \tilde{C}(y_2)$ and hence $\tilde{C}$ projects to a conjugacy between A and B.

A similar argument yields the result for cocycles of type II$'$. □

7. **Proof of Theorem 2.8.**

First we discuss cohomology of diagonal cocycles that serve as models for cocycles with two transverse invariant sub-bundles. We denote the coordinate sub-bundles by $\mathcal{E}_1$ and $\mathcal{E}_2$, and we denote a diagonal matrix with entries α_1, α_2 by diag(α_1, α_2).

Lemma 7.1. *Let $A(x) = diag(\alpha_1(x), \alpha_2(x))$, where $\alpha_{1,2}(x) \neq 0$. A preserves a measurable sub-bundle other than $\mathcal{E}_1$ and $\mathcal{E}_2$ if and only if $\alpha_1(x)/\alpha_2(x) = s(fx)/s(x)$ for a Hölder function $s(x)$, equivalently, $A(x)$ is Hölder cohomologous to $\alpha_1(x)Id$.*

Proof. Let $\mathcal{V}$ be the measurable invariant sub-bundle. Since the set where $\mathcal{V}$ differs from $\mathcal{E}_1$ and $\mathcal{E}_2$ is invariant, it is of full measure by ergodicity. Therefore we can write $\mathcal{V}(x)$ as the span of $\mathbf{v}(x) = \left[\begin{smallmatrix} s(x) \\ 1 \end{smallmatrix}\right]$, where s is a non-zero measurable function. Then $\mathbf{v}(fx) = c(x) \cdot A(x)\mathbf{v}(x)$ for a scalar function c, which implies that $\alpha_1(x)/\alpha_2(x) = s(fx)/s(x)$ and hence s is Hölder. It follows that

$$\alpha_1(x)\,\mathrm{Id} = C_A(fx) \cdot A(x) \cdot C_A(x)^{-1}, \quad \text{where} \quad C_A(x) = \mathrm{diag}\,(1, s(x)).$$

Clearly, if A is Hölder cohomologous to a scalar cocycle, then A preserves infinitely many Hölder continuous sub-bundles. □

Cocycles cohomologous to scalar ones were discussed in the previous sections.

Proposition 7. *Suppose $A(x) = diag(\alpha_1(x), \alpha_2(x))$ and $B(x) = diag(\beta_1(x), \beta_2(x))$, where $\alpha_{1,2}(x) \neq 0$ and $\beta_{1,2}(x) \neq 0$, are not cohomologous to scalar cocycles. Then*

(i) A and B are Hölder cohomologous if and only if there exist measurable, equivalently Hölder, functions s_1 and s_2 such that either
$\alpha_1(x)/\beta_1(x) = s_1(fx)/s_1(x)$ and $\alpha_2(x)/\beta_2(x) = s_2(fx)/s_2(x)$ for all x, or
$\alpha_1(x)/\beta_2(x) = s_1(fx)/s_1(x)$ and $\alpha_2(x)/\beta_1(x) = s_2(fx)/s_2(x)$ for all x.

(ii) Any measurable conjugacy between A and B is Hölder and either diagonal or anti-diagonal.

(iii) The centralizer of A consists of all constant diagonal matrices.

(iv) If A and B have conjugate periodic data, then they are Hölder cohomologous.

Proof. **(i, ii, iii)** If $A(x) = C(fx)B(x)C(x)^{-1}$ for a measurable function C, then measurable sub-bundles $C(\mathcal{E}_1)$ and $C(\mathcal{E}_2)$ are A-invariant. It follows from Lemma 7.1 that either $C(\mathcal{E}_1) = \mathcal{E}_1$ and $C(\mathcal{E}_2) = \mathcal{E}_2$, or $C(\mathcal{E}_1) = \mathcal{E}_2$ and $C(\mathcal{E}_2) = \mathcal{E}_1$. Therefore, $C(x)$ is either a diagonal or an anti-diagonal matrix. This reduces the questions of cohomology of A and B to that of cohomology of the scalar functions α_i and β_i, and (i), (ii), and (iii) follow easily.

(iv) By (ii) it suffices to show that either

$$\begin{aligned} &\alpha_1^\times(p,n)=\beta_1^\times(p,n) \ \text{ and } \ \alpha_2^\times(p,n)=\beta_2^\times(p,n) \ \text{ whenever } f^np=p, \ \text{ or} \\ &\alpha_1^\times(p,n)=\beta_2^\times(p,n) \ \text{ and } \ \alpha_2^\times(p,n)=\beta_1^\times(p,n) \ \text{ whenever } \ f^np=p. \end{aligned} \tag{14}$$

At every point $p=f^np$ the eigenvalues of $\mathcal{A}(p,n)$ and $\mathcal{B}(p,n)$ are equal, i.e.

$$\{\,\alpha_1^\times(p,n),\ \alpha_2^\times(p,n)\,\}=\{\,\beta_1^\times(p,n),\ \beta_2^\times(p,n)\,\}. \tag{15}$$

Suppose that for points $p_1=f^{n_1}p_1$ and $p_2=f^{n_2}p_2$ we have

$$\alpha_1^\times(p_1,n_1)=\beta_1^\times(p_1,n_1)\neq\beta_2^\times(p_1,n_1) \quad \text{and} \quad \alpha_1^\times(p_2,n_2)=\beta_2^\times(p_2,n_2)\neq\beta_1^\times(p_2,n_2).$$

We proceed as in the proof of Lemma 6.1. We consider orbit segments as in (11) and a periodic point $q=f^nq$ satisfying (12). By Corollary 2 there exist constants γ and σ independent of k such that

$$e^{-\gamma\epsilon^\sigma}\le\frac{\alpha_1^\times(q,kn_1)}{\alpha_1^\times(p_1,kn_1)}\le e^{\gamma\epsilon^\sigma}, \qquad e^{-\gamma\epsilon^\sigma}\le\frac{\alpha_1^\times(f^{kn_1+M_\epsilon}q,kn_2)}{\alpha_1^\times(p_2,kn_2)}\le e^{\gamma\epsilon^\sigma},$$

and the same estimates hold with β_1 in place of α_1. Taking a sufficiently large k ensures that $\alpha_1^\times(q,m)\neq\beta_1^\times(q,m)$. A similar argument shows that $\alpha_1^\times(q,m)\neq\beta_2^\times(q,m)$. This contradicts (15) and hence (14) is satisfied. □

Now we complete the proof of the theorem. We consider a cocycle A with two Hölder continuous transverse sub-bundles $\mathcal{E}_A^1$ and $\mathcal{E}_A^2$. Example (iii) in Section 8.1 shows that the sub-bundles are not necessarily orientable. Clearly, if one of the two invariant sub-bundles is orientable, then so is the other one.

(i) If the A-invariant sub-bundles are orientable, then there exist continuous unit vector fields $\mathbf{v}^1$ and $\mathbf{v}^2$ spanning $\mathcal{E}_A^1$ and $\mathcal{E}_A^2$ respectively. Let $C_A(x)$ be the change of basis matrix from $\{\mathbf{v}^1(x),\mathbf{v}^2(x)\}$ to the standard basis. Then C_A is Hölder continuous and the cocycle $A'(x)=C_A(fx)A(x)C_A(x)^{-1}$ is diagonal.

If $\mathcal{E}_A^1$ is not orientable, using a double cover as in Lemma 4.2 we obtain as in the proof of Theorem 2.2 a cocycle A'' with two transverse orientable invariant sub-bundles such that the lifts of A and A'' are Hölder cohomologous. Then A'' is cohomologous to a diagonal cocycle as before.

Now (ii) and (iii) follow from (i) and Proposition 7 as in the proofs of Theorems 2.3 and 2.6 respectively. We note that conjugacy of the periodic data implies conjugacy of the model diagonal cocycles and precludes having invariant sub-bundles of different types as in Lemma 6.3. □

8. **Examples.**

8.1. **Orientation-preserving cocycles with non-orientable invariant sub-bundles.** *There exists a smooth orientation-preserving cocycle A such that*

(i) A has a unique invariant sub-bundle that is not orientable;
(ii) A preserves infinitely many non-orientable sub-bundles;
(iii) A preserves exactly two transverse sub-bundles, which are non-orientable.

Let $\mathbb{T}^2=\mathbb{R}^2/\mathbb{Z}^2$ be the standard torus and let $\tilde{\mathbb{T}}^2=\mathbb{R}^2/(2\mathbb{Z}\times\mathbb{Z})$ be its double cover. We consider $F=\begin{bmatrix}5&2\\2&1\end{bmatrix}$, or any hyperbolic matrix $[F_{ij}]$ in $SL(2,\mathbb{Z})$ such that F_{11} is odd and F_{12} is even. The map $F:\mathbb{R}^2\to\mathbb{R}^2$ projects to Anosov automorphisms $f:\mathbb{T}^2\to\mathbb{T}^2$ and $\tilde f:\tilde{\mathbb{T}}^2\to\tilde{\mathbb{T}}^2$. Let $\tilde C:\tilde{\mathbb{T}}^2\to GL(2,\mathbb{R})$ be the function given by

$\tilde{C}(x) = R(\pi x_1)$, the rotation by the angle πx_1. This function is not 1-periodic in x_1, and hence it does not project to $\mathbb{T}^2$.

(i) We define a cocycle $\tilde{A} : \tilde{\mathbb{T}}^2 \to GL(2, \mathbb{R})$ over $\tilde{f}$ as

$$\tilde{A}(x) = \tilde{C}(\tilde{f}x)\, B\, \tilde{C}(x)^{-1}, \quad \text{where } B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \tag{16}$$

The calculation below shows that $\tilde{A}$ is 1-periodic in both x_1 and x_2 and thus it projects to a continuous and, in fact, analytic cocycle A over f on $\mathbb{T}^2$.

$$\tilde{A}(x) = R(\pi(5x_1 + 2x_2)) \left(\mathrm{Id} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) R(-\pi x_1) = R(2\pi(2x_1 + x_2)) +$$

$$+ \begin{bmatrix} \frac{\sin(2\pi(2x_1+x_2))+\sin(2\pi(3x_1+x_2))}{2} & \frac{\cos(2\pi(2x_1+x_2))+\cos(2\pi(3x_1+x_2))}{2} \\ \frac{\cos(2\pi(3x_1+x_2))-\cos(2\pi(2x_1+x_2))}{2} & \frac{\sin(2\pi(2x_1+x_2))+\sin(2\pi(3x_1+x_2))}{2} \end{bmatrix}.$$

The constant cocycle $B : \tilde{\mathbb{T}}^2 \to GL(2, \mathbb{R})$ preserves only the sub-bundle $\tilde{\mathcal{E}}_1$ spanned by the the first coordinate vector. Hence $\tilde{\mathcal{V}}(x) = \tilde{C}(x)\, \tilde{\mathcal{E}}_1$ is the unique invariant sub-bundle for $\tilde{A}$. As $\tilde{C}(x) = R(\pi x_1)$, it is easy to see that $\tilde{\mathcal{V}}$ projects to a continuous A-invariant sub-bundle $\mathcal{V}$ on $\mathbb{T}^2$, which is not orientable as its orientation is reversed along the first coordinate loop.

Clearly, the cocycles A and B on $\mathbb{T}^2$ are not continuously cohomologous as a continuous conjugacy preserves orientablility of invariant sub-bundles. In fact, they are not even measurably cohomologous, as follows from Theorem 2.3. However, their lifts are smoothly cohomologous via $\tilde{C}$ on $\tilde{\mathbb{T}}^2$.

(ii) Considering $B = \mathrm{Id}$ in (16) yields an example. Since any constant sub-bundle $\tilde{\mathcal{V}}_{\text{const}}$ is preserved by B, $\tilde{A}$ preserves the sub-bundles $\tilde{\mathcal{V}} = \tilde{C}\tilde{\mathcal{V}}_{\text{const}}$. These sub-bundles project to A-invariant non-orientable sub-bundles on $\mathbb{T}^2$.

(iii) We consider $B = \mathrm{diag}\,(2, 1)$ in (16). It is easy to see that then the cocycle A has exactly two transverse invariant sub-bundles that are non-orientable.

8.2. **Construction of Example 2.5.**

Let α and β be Hölder functions such that $\alpha(x) > 0$ and $\beta(x) > 0$ for all x in $\mathcal{M}$; for two periodic points p_1 and p_2 of periods n_1 and n_2 respectively,

$$\alpha(f^i p_1) = \beta(f^i p_1), \;\; 0 \le i \le n_1 - 1, \quad \text{and} \quad \alpha(f^i p_1) = 2\beta(f^i p_2), \;\; 0 \le i \le n_2 - 1;$$

and $\beta(x) \le \alpha(x) \le 2\beta(x) < \epsilon$ for all x. The function β can be chosen constant.

Since $\alpha^+(p, n) > 0$ and $\beta^+(p, n) > 0$ at every periodic point p, the functions α and β are not cohomologous to 0, and the matrices $\mathcal{A}(p, n)$ and $\mathcal{B}(p, n)$ are conjugate by

$$C(p) = \begin{bmatrix} \alpha^+(p, n)/\beta^+(p, n) & 0 \\ 0 & 1 \end{bmatrix}.$$

Since $1 \le \alpha^+(p, n)/\beta^+(p, n) \le 2$ for every p, $C(p)$ is uniformly bounded.

As $\alpha^+(p_1, n_1) = \beta^+(p_1, n_1)$ and $\alpha^+(p_2, n_2) = 2\beta^+(p_2, n_2)$, there is no constant c such that $\alpha^+(p, n) - c\beta^+(p, n) = 0$ for every periodic p. Thus by Proposition 3 (ii) the cocycles A and B are not measurably cohomologous.

8.3. **Construction of Examples 2.9.**

(i) We describe a simplified version of the example in Section 9 of [12].
Let $f : \mathcal{M} \to \mathcal{M}$ be a C^2 Anosov diffeomorphism that fixes a point x_0, and let $\alpha(x)$

be a smooth function such that $\alpha(x_0) = 1$ and $0 < \alpha(x) < 1$ for all $x \neq x_0$. The cocycles can be made arbitrarily close to the identity by choosing β close to 0 and $\alpha(x)$ close to 1. Since the matrices $A(x_0)$ and $B(x_0)$ are not conjugate, the cocycles A and B are not continuously cohomologous.

A measurable conjugacy is constructed in the form $C(x) = \begin{bmatrix} 1 & c(x) \\ 0 & 1 \end{bmatrix}$. Then $A(x) = C(fx)B(x)C(x)^{-1}$ is equivalent to

$$\begin{bmatrix} \alpha(x) & \beta \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha(x) & c(fx) - \alpha(x)c(x) \\ 0 & 1 \end{bmatrix}.$$

A function c such that $c(fx) = \beta + \alpha(x)c(x)$ is obtained as a series. Let

$$c_m(x) = \beta \cdot \left(1 + \alpha(f^{-1}x) + \alpha(f^{-1}x)\alpha(f^{-2}x) + \cdots + \alpha(f^{-1}x)\cdots\alpha(f^{-m}x)\right).$$

By the Birkhoff Ergodic Theorem, $(\alpha(f^{-1}x)\cdots\alpha(f^{-m}(x))^{1/m} \to \alpha < 1$ a.e. It follows that the sequence $\{c_m(x)\}$ converges to a limit $c(x)$ a.e., and the function $c(x)$ is measurable as a limit of continuous functions. The functions c_m satisfy the equation $c_m(fx) = \beta + \alpha(x)c_{m-1}(x)$, and passing to the limit we see that $c(fx) = \beta + \alpha(x)c(x)$.

(ii) Let f and α be as in (i). Clearly, B preserves the coordinate sub-bundles $\mathcal{E}_1$ and $\mathcal{E}_2$. Hence A preserves $\mathcal{E}_1$ and a measurable sub-bundle $\mathcal{V} = C\mathcal{E}_2 \neq \mathcal{E}_1$, which is not continuous. Indeed, as we show below $\mathcal{E}_1$ is the only continuous A-invariant sub-bundle. A direct calculation shows that for $p = f^n p$,

$$\mathcal{A}(p,n) = \begin{bmatrix} \alpha^\times(p,n) & \beta\cdot\alpha^*(p,n) \\ 0 & 1 \end{bmatrix}, \quad \text{where}$$
$$\alpha^*(p,n) = 1 + \alpha(f^{n-1}p) + \alpha(f^{n-1}p)\alpha(f^{n-2}p) + \cdots + \alpha(f^{n-1}p)\cdots\alpha(fp) =$$
$$= 1 + \alpha(f^{-1}p) + \alpha(f^{-1}p)\alpha(f^{-2}p) + \cdots + \alpha(f^{-1}p)\cdots\alpha(f^{-n+1}p).$$

Hence the eigenvectors of the matrix $\mathcal{A}(p,n)$ are $\mathbf{e}_1$ and

$$\mathbf{v}(p) = \begin{bmatrix} c(p) \\ 1 \end{bmatrix}, \quad \text{where} \quad c(p) = \frac{\beta\cdot\alpha^*(p,n)}{1-\alpha^\times(p,n)}. \tag{17}$$

We note that $c(p) = \lim_{m\to\infty} c_m(p)$, where $c_m(p)$ are as in (i).

Lemma 8.1. *Let $x \neq x_0$, $\epsilon > 0$, and $N > 0$. Then there exists a periodic point $q \neq x_0$ such that $dist(q,x) < \epsilon$ and $c(q) > N$.*

Proof. We assume that $0 < \epsilon < \frac{1}{2}\text{dist}(x,x_0)$ and apply the Specification Property to the orbit segments $\{x\}$ and $\{x_0, fx_0, \dots, f^{k-1}x_0\} = \{x_0,\dots,x_0\}$. Then there exists a number M_ϵ independent of k and a periodic point q such that

$$\text{dist}(q,x) < \epsilon, \quad \text{dist}(f^{M_\epsilon+1+i}q, x_0) \le \epsilon, \;\; i = 0,\dots,k-1, \quad \text{and} \;\; f^{2M_\epsilon+k+1}q = q.$$

Clearly, $q \neq x_0$. Let $q' = f^{M_\epsilon+1}q$. Since the function α is Lipschitz and $\alpha(x_0) = 1$, it follows from Corollary 2 that there exists a constant γ independent of k such that

$$\alpha(q')\alpha(fq')\dots\alpha(f^jq') \ge e^{-\gamma\epsilon} \quad \text{for} \;\; j = 0,\dots,k-1.$$

It follows that

$$c(q)/\beta \;\ge\; \alpha^*(q, 2M_\epsilon+k+1) \;\ge$$
$$\alpha(f^{2M_\epsilon+k}q)\dots\alpha(f^{M_\epsilon+k}q) + \cdots + \alpha(f^{2M_\epsilon+k}q)\dots\alpha(f^{M_\epsilon+1}q) =$$
$$\alpha(f^{2M_\epsilon+k}q)\dots\alpha(f^{M_\epsilon+k+1}q)\cdot\left(\alpha(f^{k-1}q') + \cdots + \alpha(f^{k-1}q')\cdots\alpha(q')\right) \ge m^{M_\epsilon}k\,e^{-\gamma\epsilon},$$

where $m = \min_{\mathcal{M}} \alpha(x)$. Taking a sufficiently large k ensures that $c(q) > N$. □

Let $\mathcal{V} \neq \mathcal{E}_1$ be a continuous A-invariant sub-bundle and let $x \neq x_0$ be a point such that $\mathcal{V}(x) \neq \mathcal{E}_1(x)$. Then for every periodic point p in a small neighborhood of x, $\mathcal{V}(p) \neq \mathcal{E}_1(p)$ and hence $\mathcal{V}(p)$ is spanned by $\mathbf{v}(p)$ as in (17). It follows from Lemma 8.1 and continuity of $\mathcal{V}$ that $\mathcal{V}(x) = \mathcal{E}_1(x)$, a contradiction.

(iii) We describe an example similar to one in [2]. Let $f : \mathcal{M} \to \mathcal{M}$ be an Anosov diffeomorphism and let S be a closed f-invariant set in $\mathcal{M}$ that does not contain periodic points. Let α be a smooth function such that

$$\alpha(x) = 1 \text{ for } x \in S \quad \text{and} \quad 0 < \alpha(x) < 1 \text{ for } x \notin S.$$

At every periodic point $p = f^n p$ the matrices $\mathcal{A}(p, n)$ and $\mathcal{B}(p, n)$ have the same eigenvalues, 1 and $\alpha^{\times}(p, n) < 1$, are hence are conjugate.

However, there is no continuous function C such that $A(x) = C(fx)B(x)C^{-1}(x)$. Otherwise, for $x \in S$

$$\mathcal{A}(x, n) = \begin{bmatrix} 1 & n\beta \\ 0 & 1 \end{bmatrix} = C(f^n x)\mathcal{B}(x, n)C(x)^{-1} = C(f^n x)C(x)^{-1},$$

which implies that C is unbounded.

It can be shown as in (i) that the cocycles A and B are measurably comologous. It can also be seen as in (ii) that the set of conjugacies $C(p)$ at the periodic points is unbounded, unlike in our Example 2.5.

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