PERIODIC DATA CONJUGACY FOR DIFFEOMORPHISM COCYCLES

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Abstract. We consider Hölder continuous cocycles over a hyperbolic dynamical system with values in the group of diffeomorphisms of a compact manifold. We focus on establishing conjugacy between two cocycles from conjugacy of their periodic data. We obtain such a result under bunching, or domination, assumption on the cocycles and a mild regularity assumption in C^0 topology on the maps conjugating the periodic data. We also obtain conjugacy to a constant cocycle assuming boundedness in C^1 or pre-compactness in C^0 for the constant cocycle and the conjugating maps.

1. Introduction and statements of the results

In this paper we consider diffeomorphism-valued cocycles over a hyperbolic dynamical system. We focus on the problem of obtaining conjugacy between cocycles from conjugacy of their periodic data.

By a hyperbolic system we mean either a transitive Anosov diffeomorphism of a compact connected manifold, or a topologically mixing diffeomorphism of a locally maximal hyperbolic set, or a mixing subshift of finite type. We study Hölder continuous cocycles with values in the group $\mathrm{Diff}^q(\mathcal{M})$ of C^q diffeomorphisms of a compact connected manifold $\mathcal{M}$. Hölder regularity is needed to obtain results even for scalar cocycles, and it allows symbolic systems in the base, which correspond to random and Markovian sequences of diffeomorphisms.

Let (X, f) be a hyperbolic system and let $A : X \to \mathrm{Diff}^q(\mathcal{M})$. The $\mathrm{Diff}^q(\mathcal{M})$-valued *cocycle* over f generated by A is the map

$$\mathcal{A} : X \times \mathbb{Z} \to \mathrm{Diff}^q(\mathcal{M}) \text{ defined by } \mathcal{A}(x,0) = \mathcal{A}_x^0 = \mathrm{Id}, \text{ and for } n \in \mathbb{N},$$
$$\mathcal{A}(x,n) = \mathcal{A}_x^n = A(f^{n-1}x) \circ \cdots \circ A(x) \quad \text{and} \quad \mathcal{A}(x,-n) = \mathcal{A}_x^{-n} = (\mathcal{A}_{f^{-n}x}^n)^{-1}.$$

We say that a cocycle $\mathcal{A}$ is *Hölder continuous* or *bounded* if the property holds for its generator $A(x) = \mathcal{A}_x$.

A function $\Psi : X \to \mathrm{Diff}^t(\mathcal{M})$ is called a *conjugacy* between cocycles $\mathcal{A}$ and $\mathcal{B}$ if

$$\mathcal{A}_x = \Psi_{fx} \circ \mathcal{B}_x \circ \Psi_x^{-1} \quad \text{for every } x \in X.$$

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Clearly, conjugacy of the cocycles implies *conjugacy of their periodic data*, which means that for every $p = f^n(p)$ there exists a diffeomorphism Φ_p such that

$$\mathcal{A}_p^n = \Phi_p \circ \mathcal{B}_p^n \circ \Phi_p^{-1}.$$

It is natural to ask whether conjugacy of the periodic data implies conjugacy of the cocycles. This question for Hölder continuous group-valued cocycles has been extensively studied starting with the work of Livšic [Liv71, Liv72]. In these papers, a positive answer was obtained for cocycles with values in $\mathbb{R}$ and in abelian groups. For such groups the question reduces to the case when $\mathcal{B}_x = \mathrm{Id}$, and periodic data conjugacy becomes the equality $\mathcal{A}_p^n = \mathrm{Id}$. The non-abelian case is much more difficult. The case when $\mathcal{B} = \mathrm{Id}$ has been studied the most and by now is resolved for groups with bi-invariant metrics [Liv72], $GL(d, \mathbb{R})$ [K11], Banach rings [GG17], and groups of diffeomorphism of a compact manifold in low dimension [KP16] and in arbitrary dimension [AKL18].

For non-abelian groups, the problem for two arbitrary cocycles does not reduce to the special case $\mathcal{B} = \mathrm{Id}$. In general the answer is negative even for $GL(2, \mathbb{R})$-valued cocycles with bounded periodic data conjugacy [S13]. Therefore some extra assumptions on the cocycles or on the maps Φ_p are needed. The case of a stronger assumption of equality of the periodic data $\mathcal{A}_p^n = \mathcal{B}_p^n$, that is, $\Phi_p = \mathrm{Id}$, is better understood. Cocycles in compact groups and cocycles with "bounded distortion" were studied by Parry [Pa99] and Schmidt [Sch99], who obtained positive answers for the equality case and some results for the conjugacy case. The most general results for $GL(d, \mathbb{R})$ were established in [S15] for fiber bunched cocycles and Hölder continuity of Φ_p at one point, and in [KS26] with no or weaker assumptions of Φ_p for broad classes of constant cocycles $\mathcal{B}$.

We consider cocycles with values in a group of diffeomorphism of a compact manifold. In this setting, the only result so far was obtained in [BK16] for cocycles with *equal* periodic data satisfying a certain bunching, or domination, assumption. This assumption means that the growth of the cocycle is dominated by the hyperbolicity in the base. It is used in most results for non-commutative cocycles to obtain convergence of certain iterates of the cocycle, and it yields existence of cocycle holonomies. Unlike [BK16], we study cocycles with *conjugate* periodic data. For general bunched cocycles $\mathcal{A}$ and $\mathcal{B}$, we construct a conjugacy under the assumption of Hölder continuity of Φ_p at one point in C^0 topology. We also obtain a result for a bunched cocycle $\mathcal{A}$ and a constant cocycle $\mathcal{B}$ under certain pre-compactness assumptions on $\mathcal{B}$ and Φ_p in C^0 topology. Our assumptions on Φ_p are much weaker than equality of the periodic data.

A particular difficulty in the diffeomorphism case is that the composition operation in $\mathrm{Diff}^q(\mathcal{M})$ is not continuous in C^q topology. This results in loss of regularity, which is two derivatives in [BK16]. In this paper, we obtain conjugating diffeomorphisms Ψ_x almost as regular as the cocycles.

Now we formulate the results. We denote the space of homeomorphisms of $\mathcal{M}$ by $\mathrm{Homeo}(\mathcal{M})$ and the distance between homeomorphisms by d_0. We denote the usual C^r norm adapted to the manifold setting by $|.|_{C^r}$, see Section 2.2, where a distance d_{C^r}

between diffeomorphisms is also defined. In the results below, $0 < \lambda < 1$ is so that λ and λ^{-1} are guaranteed contraction and expansion rates in the base system (X, f).

Theorem 1.1. *Let (X, f) be a hyperbolic system and let $\mathcal{M}$ be a compact connected manifold. Let $q = k + \gamma$, where $k \in \mathbb{N} = \{1, 2, \dots\}$ and $0 < \gamma < 1$, and let $k < r < q$.*

Let $\mathcal{A}$ and $\mathcal{B}$ be β-Hölder continuous Diff$^q(\mathcal{M})$-valued cocycles over (X, f) satisfying the following bunching condition with λ as in (2.1):

$$\begin{aligned} &\text{there exist } K > 0 \text{ and } \sigma > 1 \text{ such that } \sigma^{2(1+r)}\lambda^{\beta(q-r)} < 1, \text{ and} \\ &|\mathcal{A}_x^n|_{C^r} \le K\sigma^n \quad \text{and} \quad |\mathcal{B}_x^n|_{C^r} \le K\sigma^n \quad \text{for all } x \in X \text{ and } n \in \mathbb{N}. \end{aligned} \tag{1.1}$$

Suppose that for each periodic point $p = f^n(p)$ there exists $\Phi_p \in Homeo(\mathcal{M})$ such that

$$\mathcal{A}_p^n = \Phi_p \circ \mathcal{B}_p^n \circ \Phi_p^{-1}, \tag{1.2}$$

and for some periodic point p_0 we have $\Phi_{p_0} \in Diff^q(\mathcal{M})$ and for some constant c

$$d_{C^0}(\Phi_p, \Phi_{p_0}) \le c\, dist(p, p_0)^\beta \quad \text{for every periodic point } p \text{ of } f \text{ close to } p_0. \tag{1.3}$$

Then for any t such that $k < t < r$ there exists a Hölder continuous conjugacy $\Psi : X \to Diff^t(\mathcal{M})$ between $\mathcal{A}$ and $\mathcal{B}$ with $\Psi_{p_0} = \Phi_{p_0}$.

We note that Ψ_p does not necessarily coincide with Φ_p for $p \neq p_0$. For example, for $\mathcal{B} = \mathrm{Id}$ and any $\mathcal{A}$ of the form $\mathcal{A}_x = \Psi_{fx} \circ \Psi_x^{-1}$, we can take $\Phi_p = \mathrm{Id}$ at every p.

The Hölder exponent of $\Psi : X \to \mathrm{Diff}^t(\mathcal{M})$ may be smaller than β. In the proof we obtain the exponent $\beta(q-r)(r-t)^2(q-t)$. In the case of $\Phi_{p_0} = \mathrm{Id}$ we get $\beta(q-r)(r-t)$.

The bunching condition in the theorem ensures that the cocycles $\mathcal{A}$ and $\mathcal{B}$ have stable and unstable holonomies in $\mathrm{Diff}^r(\mathcal{M})$. The holonomies are defined and their properties are stated in Proposition 2.6. They play a key role in the construction of the conjugacy.

Now we state two results for a *constant* cocycle $\mathcal{B}$.

Theorem 1.2. *Let (X, f) be a hyperbolic system with a fixed point, let $\mathcal{M}$ be a compact connected manifold, let $q = k + \gamma$ where $k \in \mathbb{N}$ and $0 < \gamma < 1$, and let $k < t < q$.*

- **(1)** *Let $\mathcal{B} \equiv B$ be a constant Diff$^q(\mathcal{M})$-valued cocycle over (X, f) such that the set $\{B^n : n \in \mathbb{Z}\}$ is pre-compact in $C^0(\mathcal{M})$.*
- **(2)** *Let $\mathcal{A}$ be a β-Hölder continuous Diff$^q(\mathcal{M})$-valued cocycle over (X, f) satisfying the following bunching condition with λ as in (2.1): there exist $K > 0$, $\sigma > 1$ and $t < r < q$ such that $\sigma^{2(1+r)}\lambda^{\beta(q-r)} < 1$ and*
$$|\mathcal{A}_x^n|_{C^r} \le K\sigma^n \quad \text{for all } x \in X \text{ and } n \in \mathbb{N}.$$
- **(3)** *Suppose that for each periodic point $p = f^n(p)$ there exists $\Phi_p \in Diff^q(\mathcal{M})$ such that $\mathcal{A}_p^n = \Phi_p \circ \mathcal{B}_p^n \circ \Phi_p^{-1}$, and the set $\{\Phi_p^{-1} : p = f^n(p),\ p \in X,\ n \in \mathbb{N}\}$ is pre-compact in $C^0(\mathcal{M})$.*

Then there exists a Hölder continuous conjugacy $\Psi : X \to Diff^t(\mathcal{M})$ between $\mathcal{A}$ and $\mathcal{B}$.

This result holds with the assumptions $(1, 2, 3)$ replaced by $(1', 2', 3')$ below.

Theorem 1.3. *Let (X, f) be a hyperbolic system with a fixed point, let $\mathcal{M}$ be a compact connected manifold, let $q = k + \gamma$ where $k \in \mathbb{N}$ and $0 < \gamma < 1$, and let $k < t < q$.*

- **(1′)** *Let $\mathcal{B} \equiv B$ be a constant $Diff^q(\mathcal{M})$-valued cocycle over (X, f) such that the set $\{B^n : n \in \mathbb{N}\}$ is bounded in $Diff^1(\mathcal{M})$.*
- **(2′)** *Let $\mathcal{A}$ be a bounded $Diff^{k+1}(\mathcal{M})$-valued cocycle over (X, f) that is β-Hölder continuous as a $Diff^q(\mathcal{M})$-valued cocycle.*
- **(3′)** *Suppose that for each periodic point $p = f^n(p) \in X$ there exists $\Phi(p) \in Diff^q(\mathcal{M})$ such that $\mathcal{A}^n_p = \Phi_p \circ \mathcal{B}^n_p \circ \Phi_p^{-1}$, and the set $\{\Phi_p : p = f^n(p),\ p \in X,\ n \in \mathbb{N}\}$ is bounded in $Diff^1(\mathcal{M})$.*

Then there exists a Hölder continuous conjugacy $\Psi : X \to Diff^t(\mathcal{M})$ between $\mathcal{A}$ and $\mathcal{B}$.

We note that having a set bounded in $\mathrm{Diff}^1(\mathcal{M})$ is equivalent to having the first derivatives of the maps and their inverses uniformly bounded.

The assumption in $(2')$ that the cocycle $\mathcal{A}$ is bounded in $\mathrm{Diff}^{k+1}(\mathcal{M})$ is used only in Proposition 4.1 to obtain bunching from boundedness of the periodic data in $\mathrm{Diff}^1(\mathcal{M})$.

In Theorems 1.2 and 1.3 we obtain the Hölder exponent $\beta(q-r)(r-t)(q-t)$ for Ψ .

The paper is structured as follows. Section 2 is devoted to the preliminaries: in subsection 2.1 we describe hyperbolic dynamical systems, in 2.2 we define distance in the spaces of homeomorphisms and diffeomorphisms, and in 2.3 we give various estimates for the distances, and in 2.4 we formulate the definition and properties of the holonomies of diffeomorphism-valued cocycles. In Section 3 we prove Theorem 1.1, and in Section 4 we prove Theorems 1.2 and 1.3.

The proofs of the three theorems follow the same general scheme. First we adjust the cocycle $\mathcal{B}$ so that $\mathcal{A}$ and $\mathcal{B}$ have the same value at a fixed point p_0. Using holonomies of the cocycles, we define conjugacies Ψ^s and Ψ^u on the stable and unstable leaves of p_0. Then we show that Ψ^s and Ψ^u coincide on the homoclinic set of p_0, and thus obtain a conjugacy Ψ on this set. The main difference between the proofs of the theorems is in this step. We show that Ψ is Hölder continuous on the homoclinic set, which is dense in X, and hence Ψ extends to a Hölder continuous conjugacy between $\mathcal{A}$ and $\mathcal{B}$ on X. In Theorem 1.1, we then deduce the general case of a periodic point p_0. The proofs draw on ideas and techniques from [S15, S19, S22, KS26, BK16].

2. Preliminaries

2.1. **Hyperbolic dynamical systems.** See [KtH] for details.
Below we describe the three main classes of these systems.

Transitive Anosov diffeomorphisms. A diffeomorphism f of compact connected manifold X is called *Anosov* if there exist a splitting of the tangent bundle TX into a direct sum of two Df-invariant continuous subbundles E^s and E^u, a Riemannian

metric on X, and a number λ such that

$$\|D_x f(v^s)\| < \lambda < 1 < \lambda^{-1} < \|D_x f(v^u)\| \tag{2.1}$$

for any $x \in X$ and unit vectors $v^s \in E^s(x)$ and $v^u \in E^u(x)$. The *stable and unstable subbundles* E^s and E^u are tangent to the *stable and unstable foliations* W^s and W^u. The *local stable manifold* of x, $W^s_{loc}(x)$, is a small ball in $W^s(x)$ centered at x so that

$$\text{dist}(f^n x, f^n y) \le \lambda^n \text{dist}(x, y) \quad \text{for all } y \in W^s_{loc}(x) \text{ and } n \in \mathbb{N}.$$

Local unstable manifolds are defined similarly. We assume that they are small enough so that $W^s_{loc}(x) \cap W^u_{loc}(z)$ consists of a single point for any sufficiently close $x, z \in X$.

A diffeomorphism is said to be *(topologically) transitive* if there is a point x in X with dense orbit. All known examples of Anosov diffeomorphisms have this property. Any transitive Anosov diffeomorphism is *topological mixing*, that is, for any open sets $U_1, U_2 \subset X$ there exists $N \in \mathbb{N}$ such that $f^n(U_1) \cap U_2 \neq \emptyset$ for all $n \ge N$.

Mixing diffeomorphisms of locally maximal hyperbolic sets. More generally, let f be a diffeomorphism of a manifold $\mathcal{N}$. A compact f-invariant set $X \subset \mathcal{N}$ is called *hyperbolic* if there exist a continuous Df-invariant splitting $T_X\mathcal{N} = E^s \oplus E^u$, and a Riemannian metric on an open set $U \supset X$ such that (2.1) holds for all $x \in X$. Local stable and unstable manifolds are defined similarly for any $x \in X$, and we denote their intersections with X by $W^s_{loc}(x)$ and $W^u_{loc}(x)$. The set X is called *locally maximal* if $X = \bigcap_{n\in\mathbb{Z}} f^{-n}(U)$ for some open set $U \supset X$. This property ensures that $W^s_{loc}(x) \cap W^u_{loc}(z)$ exists in X. We assume that $f|X$ is topologically mixing.

Mixing subshifts of finite type. Let M be $k \times k$ matrix with entries from $\{0, 1\}$ such that all entries of M^N are positive for some N. Let

$$X = \{\, x = (x_n)_{n\in\mathbb{Z}} : \ 1 \le x_n \le k \ \text{ and } \ M_{x_n, x_{n+1}} = 1 \ \text{ for every } n \in \mathbb{Z} \,\}.$$

The shift map $f : X \to X$ is defined by $(f(x))_n = x_{n+1}$. The system (X, f) is called a *mixing subshift of finite type*. We fix $\lambda \in (0, 1)$ and consider the metric

$$d(x, y) = d_\lambda(x, y) = \lambda^{n(x,y)}, \ \text{ where } \ n(x, y) = \min\{\, |i| \ : \ x_i \neq y_i \}.$$

The following sets play the role of the stable and unstable manifolds of x

$$W^s(x) = \{\, y : \text{ for some } i_0, \ x_i = y_i \ \text{ for all } \ i \ge i_0 \,\},$$
$$W^u(x) = \{\, y : \text{ for some } i_0, \ x_i = y_i \ \text{ for all } \ i \le i_0 \,\},$$
$$W^s_{loc}(x) = \{\, y : \ x_i = y_i \text{ for all } i \ge 0 \,\}, \quad W^u_{loc}(x) = \{\, y : \ x_i = y_i \text{ for all } i \le 0 \,\}.$$

Anosov Closing Lemma. These systems have the following property.

Lemma 2.1. [KtH, 6.4.15-17] *Let (X, f) be a hyperbolic dynamical system as above. Then there exist constants $c', \delta' > 0$ such that for any $x \in X$ and $k \in \mathbb{N}$ with $dist(x, f^k(x)) < \delta'$ there exists a periodic point $p \in X$ with $f^k p = p$ such that the orbit segments $x, f(x), \ldots, f^k(x)$ and $p, f(p), \ldots, f^k(p)$ satisfy*

$$dist(f^i(x), f^i(p) \le c\, dist(x, f^k(x)) \quad \textit{for every } i = 0, \ldots, k.$$

2.2. **Distances on the space $\mathbf{Diff}^r(\mathcal{M})$.** ([dlLW10, Section 5], [S19, Section 2.2]) We fix a smooth background Riemannian metric and the corresponding distance $d_{\mathcal{M}}$ on $\mathcal{M}$. We denote the space of homeomorphisms of $\mathcal{M}$ by $\mathrm{Homeo}(\mathcal{M})$, and for $g, h \in \mathrm{Homeo}(\mathcal{M})$ we set

$$d_0(g,h) = \max_{t\in\mathcal{M}} d_{\mathcal{M}}(g(t),h(t)) \quad \text{and} \quad d_{C^0}(g,h) = d_0(g,h) + d_0(g^{-1},h^{-1}). \tag{2.2}$$

Now we consider $r \geq 1$. The C^r topology on the group of diffeomorphisms $\mathrm{Diff}^r(\mathcal{M})$ can be defined using coordinate patches and the C^r norm in the Euclidean space. For any $g \in \mathrm{Diff}^r(\mathcal{M})$, $r \in \mathbb{N}$, we define its C^r size as

$$|g|_{C^r} = \|g\|_{C^r} + \|g^{-1}\|_{C^r}, \quad \text{where } \ \|g\|_{C^r} = \max_{t\in\mathcal{M}} d_{\mathcal{M}}(g(t),t) + \max_{1\leq i\leq r} \max_{t\in\mathcal{M}} \|D_t^i g\|.$$

Here $D_t^i g$ is the derivative of g of order i at t, and its norm is defined as the norm of the corresponding multilinear form from $T_t\mathcal{M}$ to $T_{g(t)}\mathcal{M}$ with respect the Riemannian metric. For $r = k + \alpha$, with $k \in \mathbb{N}$ and $0 < \alpha < 1$, the definition is similar with

$$\|g\|_{C^r} = \|g\|_{C^k} + \sup\{\, \|D_t^k g - D_{t'}^k g\| \cdot d_{\mathcal{M}}(t,t')^{-\alpha} :\ t,t' \in \mathcal{M},\ 0 < d_{\mathcal{M}}(t,t') < \epsilon_0\}.$$

Here ϵ_0 is chosen small compared to the injectivity radius of $\mathcal{M}$ so that the tangent bundle is locally trivialized via parallel transport and the difference makes sense.

A natural distance $d_{C^r}(g,h)$ on $\mathrm{Diff}^r(\mathcal{M})$, $r \in \mathbb{N}$, was defined in [dlLW10] as the infimum of the lengths of piecewise C^1 paths in $\mathrm{Diff}^r(\mathcal{M})$ connecting h with g and h^{-1} with g^{-1}, where the length of a path p_s is

$$\ell_{C^r}(p_s) = \max_{0\leq i\leq r} \max_{t\in\mathcal{M}} \int \|\tfrac{d}{ds}(D_t^i p_s)\|\, ds.$$

For $r = k + \alpha$, one also adds the corresponding Hölder term:

$$\ell_{C^r}(p_s) = \ell_{C^k}(p_s) + \max_{t\in\mathcal{M}} \int |\tfrac{d}{ds}\|(D^k p_s)\|_{\alpha,t}|\, ds, \quad \text{where}$$

$$\|D^k g\|_{\alpha,t} = \sup\{\, \|D_{t'}^k g - D_t^k g\| \cdot d_{\mathcal{M}}(t',t)^{-\alpha} :\ t' \in \mathcal{M},\ 0 < d_{\mathcal{M}}(t,t') < \epsilon_0\}.$$

For sufficiently C^r-close diffeomorphisms, the distance $d_{C^r}(g,h)$ is Lipschitz equivalent to $\|g-h\|_{C^r} + \|g^{-1}-h^{-1}\|_{C^r}$, where the difference is understood using local trivialization. Specifically, there exist constants κ and $\delta_0 > 0$ depending only on r and the Riemannian metric so that

$$\begin{aligned} &\kappa^{-1} d_{C^r}(g,h) \leq \|g-h\|_{C^r} + \|g^{-1}-h^{-1}\|_{C^r} \leq \kappa\, d_{C^r}(g,h), \ \text{ provided that} \\ &\text{either } \ d_{C^r}(g,h) < \delta_0 |g|_{C^r}^{-1} \ \text{ or } \ \|g-h\|_{C^r} + \|g^{-1}-h^{-1}\|_{C^r} < \delta_0 |g|_{C^r}^{-1}. \end{aligned} \tag{2.3}$$

2.3. **Estimates of norms and distances.**
In the arguments we will use the following estimates for distances d_0 and d_{C^r}.

Lemma 2.2. *If the maps below are homeomorphisms of $\mathcal{M}$ and g is differentiable with $\|Dg\| \leq M_g$ then*

$$d_0(h_1 \circ g, h_2 \circ g) = d_0(h_1,h_2) \quad \textit{and} \quad d_0(g\circ h_1, g\circ h_2) \leq M_g\, d_0(h_1,h_2) \tag{2.4}$$

$$d_0(h_1 \circ g \circ h_2, g) \leq d_0(h_1, Id) + M_g\, d_0(h_2, Id) \tag{2.5}$$

$$d_0(h_1 \circ h_2, Id) \le d_0(h_1, Id) + d_0(h_2, Id), \tag{2.6}$$

$$d_0(g \circ h,\ \tilde{g} \circ \tilde{h}) \le d_0(g, \tilde{g}) + M_g\, d_0(h, \tilde{h}), \quad \text{and} \tag{2.7}$$

$$d_0(h_1 \circ h_2 \circ h_3, \tilde{h}_1 \circ \tilde{h}_2 \circ \tilde{h}_3,) \le d_0(h_1, \tilde{h}_1) + M_{h_1} d_0(h_2, \tilde{h}_2) + M_{h_1} M_{h_2} d_0(h_3, \tilde{h}_3), \tag{2.8}$$

provided that h_1 and h_2 are differentiable with $\|Dh_1\| \le M_{h_1}$ and $\|Dh_2\| \le M_{h_2}$.

Proof. The inequalities (2.5) and (2.6) are obtained, respectively, using

$$d_0(h_1 \circ g \circ h_2, g) \le d_0(h_1 \circ g \circ h_2, g \circ h_2) + d_0(g \circ h_2, g) \quad \text{and}$$
$$d_0(h_1 \circ h_2, \mathrm{Id}) \le d_0(h_1 \circ h_2, h_2) + d_0(h_2, \mathrm{Id}),$$

and (2.7), (2.8) follow similarly. □

Lemma 2.3. (Follows directly from [dlLO99, Proposition 5.5])
For any $r \ge 1$ there exists a constant M_r such that for any $h, g \in C^r(\mathcal{M})$,

$$\|h \circ g\|_{C^r} \le M_r \, \|h\|_{C^r} (1 + \|g\|_{C^r})^r.$$

Lemma 2.4. [dlLO99, Proposition 6.2(iii)]
Let $k \in \mathbb{N}$ and $k < t < r < k + 1$. There exist $0 < \delta_1 < 1$ and $M' > 0$ such that if $h_1, h_2 \in C^t(\mathcal{M})$, $\|h_1 - h_2\|_{C^t} < \delta_1$, and $g \in C^r(\mathcal{M})$, then

$$\|g \circ h_1 - g \circ h_2\|_{C^t} \le M' \, \|g\|_{C^r} \, \|h_1 - h_2\|_{C^t}^{r-t}.$$

Lemma 2.5. *Let $k \in \mathbb{N}$ and $k < t < r < k + 1$. There exists $M = M(r, t, \mathcal{M}, K)$ such that for any $g, \tilde{g} \in Diff^r(\mathcal{M})$ and $h_1, h_2 \in Diff^t(\mathcal{M})$ with $|h_1|_{C^t}, |h_2|_{C^t} \le K$, the following holds.*

$$d_{C^t}(g \circ h_1,\ g \circ h_2) \le M \left(\|g\|_{C^r} + (1 + \|g^{-1}\|_{C^t})^t \right) \cdot d_{C^t}(h_1, h_2)^{r-t}, \quad \text{and} \tag{2.9}$$

$$d_{C^t}(h_1 \circ g,\ h_2 \circ g) \le M \left(\|g^{-1}\|_{C^r} + (1 + \|g\|_{C^t})^t \right) \cdot d_{C^t}(h_1, h_2)^{r-t} \tag{2.10}$$

provided that $d_{C^t}(h_1, h_2) < \delta |h_1|_{C^t}^{-1}$, where $\delta = \min\{\delta_0, \kappa^{-1}\delta_1\}$ with δ_0, κ from (2.3) and δ_1 from Lemma 2.4, and in each case the right hand side is less than

$$\delta \big(M_t (1 + |h_1|_{C^t})^t (1 + |g|_{C^t})^t \big)^{-1}. \tag{2.11}$$

Also

$$\begin{aligned} &d_{C^t}(g \circ h_1 \circ \tilde{g},\ g \circ h_2 \circ \tilde{g}) \\ &\le M \left(\|g\|_{C^r} (1 + \|\tilde{g}\|_{C^t})^t + \|\tilde{g}^{-1}\|_{C^r} (1 + \|g^{-1}\|_{C^t})^t \right) \cdot d_{C^t}(h_1, h_2)^{r-t} \end{aligned} \tag{2.12}$$

provided that $d_{C^t}(h_1, h_2) \le \delta |h_1|_{C^t}^{-1}$ and the right hand side of (2.12) is less than

$$\delta \left(M_t^2 \, (1 + |h_1|_{C^t})^t (\|g\|_{C^t} \, (1 + \|\tilde{g}\|_{C^t})^t + \|\tilde{g}^{-1}\|_{C^t} \, (1 + \|g^{-1}\|_{C^t})^t) \right)^{-1}. \tag{2.13}$$

Proof. The last part of the lemma was proven in [S19, Lemma 3.6]. For the sake of completeness, we give a direct argument for (2.9) using a similar approach.

We use (2.3) for distances between close diffeomorphisms. Lemma 2.4 gives an estimate for $\|g \circ h_1 - g \circ h_2\|_{C^t}$. By Lemma 2.3, we have

$$\|h_1^{-1} \circ g^{-1} \ - h_2^{-1} \circ g^{-1}\|_{C^t} = \|(h_1^{-1} - h_2^{-1}) \circ g^{-1}\|_{C^t} \le M_t \|h_1^{-1} - h_2^{-1}\|_{C^t} (1 + \|g^{-1}\|_{C^t})^t.$$

Hence

$$\begin{aligned}&\|g\circ h_1 - g\circ h_2\|_{C^t} + \|h_1^{-1}\circ g^{-1} \ - h_2^{-1}\circ g^{-1}\|_{C^t} \le\\ &M'\,\|g\|_{C^r}\,\|h_1-h_2\|_{C^t}^{r-t} + M_t\,\|h_1^{-1}-h_2^{-1}\|_{C^t}(1+\|g^{-1}\|_{C^t})^t \le\\ &M''\left(\|g\|_{C^r} + (1+\|g^{-1}\|_{C^t})^t\right)\cdot d_{C^t}(h_1,h_2)^{r-t} =: (*).\end{aligned}$$

By (2.3), we also have $\kappa^{-1}d_{C^t}(h_1\circ g,\, h_2\circ g) \le (*)$, yielding (2.9) with $M = \kappa M''$, provided that $(*) < \delta_0|g\circ h_1|_{C^t}^{-1}$. To show the latter we apply Lemma 2.3 to estimate

$$\begin{aligned}\|h_1^{-1}\circ g^{-1}\|_{C^t} &\le M_t\,\|h_1^{-1}\|_{C^t}(1+\|g^{-1}\|_{C^t})^t \le M_t(1+|h_1|_{C^t})^t(1+\|g\|_{C^t})^t,\\ \|g\circ h_1\|_{C^t} &\le M_t(1+\|h_1\|_{C^t})^t\,\|g\|_{C^t}, \ \text{ and so}\\ |h_1\circ g|_{C^t} &\le M_t(1+|h_1|_{C^t})^t(1+|g|_{C^t})^t.\end{aligned}$$

Thus by the assumption (2.11) we have $(*) < \delta|h_1\circ g|_{C^t}^{-1} \le \delta_0|h_1\circ g|_{C^t}^{-1}$. □

2.4. **Holonomies.** In the proofs, a conjugacy Ψ will be constructed using holonomies of the cocycles $\mathcal{A}$ and $\mathcal{B}$. We recall the following result on existence and properties of holonomies for diffeomorphism cocycles.

Proposition 2.6. [S19, Proposition 3.3 and Remark 3.4]
Let $k\in\mathbb{N}$ and $k<r<q<k+1$, and let λ be as in (2.1).

Let $\mathcal{A}$ be a is β-Hölder $Diff^q(\mathcal{M})$-valued cocycle over a hyperbolic system (X,f). Suppose that there exist $K>0$ and $\sigma>1$ satisfying $\sigma^{2(1+r)}\lambda^{\beta(q-r)}<1$ such that

$$|\mathcal{A}_x^n|_{C^q} \le K\sigma^n \ \text{ for all } n\in\mathbb{N}. \tag{2.14}$$

Then for any $x\in X$ and $y\in W^s(x)$, the limit

$$H^s_{x,y} = H^{\mathcal{A},s}_{x,y} = \lim_{n\to\infty}(\mathcal{A}_y^n)^{-1}\circ\mathcal{A}_x^n \quad \text{exists in } Diff^r(\mathcal{M}) \text{ and satisfies} \tag{2.15}$$

- **(H1)** $H^s_{x,x} = Id$ *and* $H^s_{y,z}\circ H^s_{x,y} = H^s_{x,z}$, *which imply* $(H^s_{x,y})^{-1} = H^s_{y,x}$;
- **(H2)** $H^s_{x,y} = (\mathcal{A}_y^n)^{-1}\circ H^s_{f^nx,\,f^ny}\circ\mathcal{A}_x^n$ *for all* $n\in\mathbb{N}$;
- **(H3)** *There exists a constant c such that for all $x\in X$ and $y\in W^s_{loc}(x)$,*

$$\begin{aligned}&d_{C^r}(H^s_{x,y}, Id) \le c\,dist(x,y)^{\beta(q-r)}, \quad \text{moreover,}\\ &d_{C^r}\left((\mathcal{A}_y^n)^{-1}\circ\mathcal{A}_x^n,\ Id\right) \le c\,dist(x,y)^{\beta(q-r)} \quad \text{for all } n\in\mathbb{N}.\end{aligned} \tag{2.16}$$

The maps $H^s_{x,y}$ are called the *stable holonomies of* $\mathcal{A}$.
The *unstable holonomies* are defined similarly: for $y\in W^u(x)$,

$$H^u_{x,y} = H^{\mathcal{A},u}_{x,y} = \lim_{n\to-\infty}(\mathcal{A}_y^n)^{-1}\circ\mathcal{A}_x^n = \lim_{n\to\infty}(\mathcal{A}^n_{f^{-n}y})\circ(\mathcal{A}^n_{f^{-n}x})^{-1}.$$

They exist under the same assumptions and satisfy similar properties.

3. Proof of Theorem 1.1

3.1. Reduction to the case of $\mathcal{A}^k_{p_0} = \mathcal{B}^k_{p_0}$ and $\Phi(p_0) = \mathbf{Id}$.

Let k be the prime period of p_0. Instead of the cocycle $\mathcal{B}$ we will work with

$$\tilde{\mathcal{B}} = \Phi_{p_0} \circ \mathcal{B} \circ \Phi_{p_0}^{-1}, \quad \text{so that} \quad \tilde{\mathcal{B}}^k_{p_0} = \mathcal{A}^k_{p_0}.$$

Since Φ_{p_0} is in $\mathrm{Diff}^q(\mathcal{M})$, so is $\tilde{B}$. Since $\mathcal{B}$ is Hölder continuous with exponent β, it follows from Lemma 2.5 (2.12) that $\tilde{\mathcal{B}}$ is Hölder continuous with exponent $\beta(q-r)$.

By Proposition 2.6, the original cocycle $\mathcal{B}$ has holonomies in $\mathrm{Diff}^r(\mathcal{M})$ satisfying (H1, H2, H3). The cocycle $\tilde{\mathcal{B}}$ also has holonomies in $\mathrm{Diff}^r(\mathcal{M})$ given by

$$H^{\tilde{\mathcal{B}},s}_{x,y} = \Phi_{p_0} \circ H^{\mathcal{B},s}_{x,y} \circ \Phi_{p_0}^{-1} \quad \text{and} \quad H^{\tilde{\mathcal{B}},u}_{x,y} = \Phi_{p_0} \circ H^{\mathcal{B},u}_{x,y} \circ \Phi_{p_0}^{-1}.$$

They satisfy (H1) and (H2), but the property (H3) does not necessarily hold for $H^{\tilde{\mathcal{B}},s}$. However, by (H3) and (2.4) we have

(H$\tilde{3}$) There exists a constant $\tilde{c}$ such that for all $x \in X$ and $y \in W_{loc}(x)$

$$d_{C^0}(H^{\tilde{\mathcal{B}},s}_{x,y}, \mathrm{Id}) \le \tilde{c}\, \mathrm{dist}(x,y)^{\beta(q-r)}. \tag{3.1}$$

Also, by (H3) and Lemma 2.5 (2.12),

(H$\bar{3}$) There exists a constant $\bar{c}$ such that for all $x \in X$ and $y \in W_{loc}(x)$

$$|H^{\tilde{\mathcal{B}},s}_{x,y}|_{C^r} \le \bar{c} \quad \text{and} \quad d_{C^t}(H^{\tilde{\mathcal{B}},s}_{x,y}, \mathrm{Id}) \le \bar{c}\, \mathrm{dist}(x,y)^{\beta(q-r)(r-t)}. \tag{3.2}$$

The function $\tilde{\Phi}_p = \Phi_p \circ \Phi_{p_0}^{-1}$ is a periodic data conjugacy between $\mathcal{A}$ and $\tilde{\mathcal{B}}$ with $\tilde{\Phi}_{p_0} = \mathrm{Id}$ satisfying (1.3).

We will construct a Hölder continuous conjugacy $\tilde{\Psi} : X \to \mathrm{Diff}^t(\mathcal{M})$ between $\mathcal{A}$ and $\tilde{\mathcal{B}}$ with the exponent $\beta(q-r)(r-t)^2$. Then $\Psi = \tilde{\Psi} \circ \Phi_{p_0}$ will be a conjugacy between $\mathcal{A}$ and $\mathcal{B}$ with values in $\mathrm{Diff}^t(\mathcal{M})$, Hölder continuous with exponent $\beta(q-r)(r-t)^2(q-t)$.

If $\Phi_{p_0} = \mathrm{Id}$ for the original cocycles, the reduction is not needed and the conjugacy Ψ will be Hölder continuous with exponent $\beta(q-r)(r-t)$.

Now we construct a conjugacy $\tilde{\Psi}$ between $\mathcal{A}$ and $\tilde{\mathcal{B}}$. To simplify the notations, we will write $\mathcal{B}$ for $\tilde{\mathcal{B}}$, Φ for $\tilde{\Phi}$, and Ψ for $\tilde{\Psi}$. Thus,

In the rest of the proof of the theorem,

- $\mathcal{A}$ and $\mathcal{B}$ are $\mathrm{Diff}^q(\mathcal{M})$-valued cocyles over (X, f) with $\mathcal{B}^k_{p_0} = \mathcal{A}^k_{p_0}$;
- $\mathcal{A}$ and $\mathcal{B}$ have stable and unstable holonomies in $\mathrm{Diff}^r(\mathcal{M})$,
  the holonomies $H^{\mathcal{A},s/u}$ for $\mathcal{A}$ satisfying (H1, H2, H3),
  the holonomies $H^{\mathcal{B},s/u}$ for $\mathcal{B}$ satisfying (H1, H2, H$\tilde{3}$, H$\bar{3}$);
- Φ_p is a periodic data conjugacy between $\mathcal{A}$ and B with $\Phi_{p_0} = \mathrm{Id}$ satisfying

$$d_{C^0}(\Phi_p, \mathrm{Id}) \le c\, \mathrm{dist}(p, p_0)^\beta \quad \text{for every periodic point } p \text{ of } f \text{ close to } p_0. \tag{3.3}$$

First we assume that p_0 is a **fixed point** of f and construct a conjugacy between cocycles $\mathcal{A}$ and $\mathcal{B}$ with $\mathcal{A}_{p_0} = \mathcal{B}_{p_0}$. We remove this assumption in Section 3.4.

3.2. **Construction of a conjugacy Ψ on the homoclinic set of a fixed point.** We construct conjugacies between $\mathcal{A}$ and $\mathcal{B}$ along the stable and unstable manifolds of p_0 and show that they coincide at every homoclinic point of p_0. We set

$$\begin{aligned} \Psi^s_x &= H^{\mathcal{A},s}_{p_0,\,x} \circ H^{\mathcal{B},s}_{x,\,p_0} \quad \text{for } x \in W^s(p_0), \\ \Psi^u_x &= H^{\mathcal{A},u}_{p_0,\,x} \circ H^{\mathcal{B},u}_{x,\,p_0} \quad \text{for } x \in W^u(p_0). \end{aligned} \tag{3.4}$$

The function Ψ^s is a conjugacy between $\mathcal{A}$ and $\mathcal{B}$ along the leaf $W^s(p_0)$. Indeed, If $x \in W^s(p_0)$, then using property (H2) of the holonomies with $n = 1$ we obtain

$$\begin{aligned} &\Psi^s_{f(x)} \circ \mathcal{B}_x \circ \Psi^s(x)^{-1} \;=\; H^{\mathcal{A},s}_{f(p_0),\,f(x)} \circ H^{\mathcal{B},s}_{f(x),\,f(p_0)} \circ \mathcal{B}_x \circ (H^{\mathcal{B},s}_{x,p_0})^{-1} \circ (H^{\mathcal{A},u}_{p_0,x})^{-1} \;= \\ &H^{\mathcal{A},s}_{f(p_0),\,f(x)} \circ \mathcal{B}_{p_0} \circ (H^{\mathcal{A},u}_{p_0,x})^{-1} \;=\; H^{\mathcal{A},s}_{f(p_0),\,f(x)} \circ \mathcal{A}_{p_0} \circ (H^{\mathcal{A},u}_{p_0,x})^{-1} \;=\; \mathcal{A}_x. \end{aligned}$$

Similarly, Ψ^u is a conjugacy between $\mathcal{A}$ and $\mathcal{B}$ along the leaf $W^u(p_0)$.

Let $x \in W^s(p_0) \cap W^u(p_0)$. We will show that $\Psi^s_x = \Psi^u_x$, equivalently,

$$H^{\mathcal{B},u}_{p_0,x} \circ H^{\mathcal{A},u}_{x,p_0} \circ H^{\mathcal{A},s}_{p_0,\,x} \circ H^{\mathcal{B},s}_{x,\,p_0} = \mathrm{Id}.$$

Using the definition of the holonomies and the assumption that $\mathcal{A}_{p_0} = \mathcal{B}_{p_0}$, we obtain

$$H^{\mathcal{A},s}_{p_0,\,x} \circ H^{\mathcal{B},s}_{x,\,p_0} = \lim_{n\to\infty} \left((\mathcal{A}^n_x)^{-1} \circ \mathcal{A}^n_{p_0} \circ (\mathcal{B}^n_{p_0})^{-1} \circ \mathcal{B}^n_x\right) = \lim_{n\to\infty} \left((\mathcal{A}^n_x)^{-1} \circ \mathcal{B}^n_x\right)$$

and

$$H^{\mathcal{B},u}_{p_0,\,x} \circ H^{\mathcal{A},u}_{x,\,p_0} = \lim_{n\to\infty} \left(\mathcal{B}^n_{f^{-n}x} \circ (\mathcal{A}^n_{f^{-n}x})^{-1}\right).$$

Thus,

$$\begin{aligned} H^{\mathcal{B},u}_{p_0,\,x} \circ H^{\mathcal{A},u}_{x,\,p_0} \circ H^{\mathcal{A},s}_{p_0,\,x} \circ H^{\mathcal{B},s}_{x,\,p_0} &= \lim_{n\to\infty} \left(\mathcal{B}^n_{f^{-n}x} \circ (\mathcal{A}^n_{f^{-n}x})^{-1} \circ (\mathcal{A}^n_x)^{-1} \circ \mathcal{B}^n_x\right) = \\ &= \lim_{n\to\infty} \left(\mathcal{B}^n_{f^{-n}x} \circ (\mathcal{A}^{2n}_{f^{-n}x})^{-1} \circ \mathcal{B}^n_x\right). \end{aligned}$$

We will show that $d_0(\mathcal{B}^n_{f^{-n}x} \circ (\mathcal{A}^{2n}_{f^{-n}x})^{-1} \circ \mathcal{B}^n_x,\, \mathrm{Id}) \to 0$.

We consider an orbit segment $\{f^{-n}x, \dots, x, \dots, f^n(x)\}$. Since

$$\mathrm{dist}(f^{-n}(x), f^n(x)) \le \mathrm{dist}(f^{-n}(x), p_0) + \mathrm{dist}(f^n(x), p_0) \le K_x\lambda^n \quad \text{for all } n \in \mathbb{N},$$

for all sufficiently large n we have

$$\delta_n = \mathrm{dist}(f^{-n}(x), f^n(x)) \le K_x\lambda^n < \delta',$$

where δ' is as in the Anosov Closing Lemma 2.1. Hence there exists a periodic point $p_n = f^{2n}(p_n)$ such that such that

$$\mathrm{dist}(f^i(x), f^i(p_n)) \le c'\delta_n \quad \text{for every } i = -n, \dots, n.$$

To simplify the notations we will write p for p_n.

We will prove that

$$d_0(\mathcal{B}^n_{f^{-n}x} \circ (\mathcal{A}^{2n}_{f^{-n}x})^{-1} \circ \mathcal{B}^n_x,\ \mathcal{B}^n_{f^{-n}p} \circ (\mathcal{A}^{2n}_{f^{-n}p})^{-1} \circ \mathcal{B}^n_p) \to 0 \quad \text{and}$$
$$d_0(\mathcal{B}^n_{f^{-n}p} \circ (\mathcal{A}^{2n}_{f^{-n}p})^{-1} \circ \mathcal{B}^n_p,\ \mathrm{Id}) \to 0.$$

We write $\mathcal{B}^n_{f^{-n}x}$, $\mathcal{B}^n_x$, and $(\mathcal{A}^{2n}_{f^{-n}x})^{-1}$ in terms of the values at p. Let

$$y = y_n = W^s_{loc}(p) \cap W^u_{loc}(x).$$

Then for each $i = -n, \dots, n$,

$$\mathrm{dist}(f^i(x), f^i(y)) \le c_1\delta_n \ \text{ and } \ \mathrm{dist}(f^i(y), f^i(p)) \le c_1\delta_n.$$

By property (H2) of the holonomies we have

$$\mathcal{B}^n_x \ = \ H^{\mathcal{B},u}_{f^n y,\, f^n x} \circ \mathcal{B}^n_y \circ H^{\mathcal{B},u}_{x,y} \ = \ H^{\mathcal{B},u}_{f^n y,\, f^n x} \circ H^{\mathcal{B},s}_{f^n p,\, f^n y} \circ \mathcal{B}^n_p \circ H^{\mathcal{B},s}_{y,p} \circ H^{\mathcal{B},u}_{x,y}.$$

By property (H3), for each of the holonomies above we have

$$d_{C^0}(H, \mathrm{Id}) \le \tilde{c}(c_1\delta_n)^{\beta(q-r)} = c_2\lambda^{n\beta(q-r)},$$

and by (2.6) it follows that

$$\mathcal{B}^n_x = H_1 \circ \mathcal{B}^n_p \circ H_2, \ \text{ where } \ d_0(H_1, \mathrm{Id}) \le 2c_2\,\lambda^{n\beta(q-r)} \ \text{ and } \ d_0(H_2, \mathrm{Id}) \le 2c_2\,\lambda^{n\beta(q-r)}.$$

By the bunching assumption,

$$\|D\mathcal{B}^n_p\| \le K\sigma^n \quad \text{where } \ \sigma^{2(1+r)}\lambda^{\beta(q-r)} < 1.$$

Using (2.5) we estimate

$$\begin{aligned} d_0(\mathcal{B}^n_x,\, \mathcal{B}^n_p) = \ & d_0(H_1 \circ \mathcal{B}^n_p \circ H_2,\, \mathcal{B}^n_p) \ \le \ d_0(H_1, \mathrm{Id}) + \|D\mathcal{B}^n_p\| \cdot d_0(H_2, \mathrm{Id}) \\ & \le 2c_2\,\lambda^{n\beta(q-r)} + K\sigma^n \cdot 2c_2\,\lambda^{n\beta(q-r)} \le \ c_3\,\sigma^n\lambda^{n\beta(q-r)}. \end{aligned}$$

Similarly,

$$d_0(\mathcal{B}^n_{f^{-n}x},\, \mathcal{B}^n_{f^{-n}p}) \le c_3\,\sigma^n\lambda^{n\beta(q-r)} \quad \text{and} \quad d_0((\mathcal{A}^{2n}_{f^{-n}x})^{-1},\, (\mathcal{A}^{2n}_{f^{-n}p})^{-1}) \le c_3\,\sigma^{2n}\lambda^{n\beta(q-r)}.$$

Using (2.8) we deduce

$$\begin{aligned} & d_0(\mathcal{B}^n_{f^{-n}x} \circ (\mathcal{A}^{2n}_{f^{-n}x})^{-1} \circ \mathcal{B}^n_x,\ \mathcal{B}^n_{f^{-n}p} \circ (\mathcal{A}^{2n}_{f^{-n}p})^{-1} \circ \mathcal{B}^n_p) \le \\ & d_0(\mathcal{B}^n_{f^{-n}x}, \mathcal{B}^n_{f^{-n}p}) + \|D\mathcal{B}^n_p\|\, d_0((\mathcal{A}^{2n}_{f^{-n}x})^{-1}, (\mathcal{A}^{2n}_{f^{-n}p})^{-1}) \\ & \qquad + \|D\mathcal{B}^n_p\|\, \|D(\mathcal{A}^{2n}_{f^{-n}p})^{-1}\|\, d_0(\mathcal{B}^n_x, \mathcal{B}^n_p) \le \\ & c_3\,\sigma^n\lambda^{n\beta(q-r)} + K\sigma^n c_3\,\sigma^{2n}\lambda^{n\beta(q-r)} + K\sigma^n K\sigma^{2n} c_3\,\sigma^n\lambda^{n\beta(q-r)} \le \ c_4\,\sigma^{3n}\lambda^{n\beta(q-r)}. \end{aligned}$$

Since $r > 1$ we have $\sigma^3\lambda^{\beta(q-r)} < \sigma^{2(1+r)}\lambda^{\beta(q-r)} < 1$, and hence $\sigma^{3n}\lambda^{n\beta(q-r)} \to 0$ as $n \to \infty$.

Next we consider $\mathcal{B}^n_{f^{-n}p} \circ (\mathcal{A}^{2n}_{f^{-n}p})^{-1} \circ \mathcal{B}^n_p$. We observe that

$$(\mathcal{A}^{2n}_{f^{-n}p})^{-1} \ = \ \Phi_{f^{-n}p} \circ (\mathcal{B}^{2n}_{f^{-n}p})^{-1} \circ \Phi^{-1}_{f^{-n}p},$$

where

$$d_{C^0}\left(\Phi_{f^{-n}p}, \mathrm{Id}\right) = d_{C^0}(\Phi_{f^{-n}p}, \Phi_{p_0}) \le c\,\mathrm{dist}(f^{-n}p, p_0)^\beta \le c\lambda^{n\beta}.$$

Then using (2.5) we obtain

$$d_0((\mathcal{A}^{2n}_{f^{-n}p})^{-1},\,(\mathcal{B}^{2n}_{f^{-n}p})^{-1}) = d_0(\Phi_{f^{-n}p}\circ(\mathcal{B}^{2n}_{f^{-n}p})^{-1}\circ\Phi^{-1}_{f^{-n}p},\,(\mathcal{B}^{2n}_{f^{-n}p})^{-1}) \le$$
$$d_0(\Phi_{f^{-n}p},\mathrm{Id}) + \|D(\mathcal{B}^{2n}_{f^{-n}p})^{-1}\|\, d_0(\Phi^{-1}_{f^{-n}p},\mathrm{Id}) \le c\lambda^{n\beta} + K\sigma^{2n}c\lambda^{n\beta} \le c_5\,\sigma^{2n}\lambda^{n\beta}.$$

Finally, using (2.4) we estimate

$$d_0(\mathcal{B}^n_{f^{-n}p}\circ(\mathcal{A}^{2n}_{f^{-n}p})^{-1}\circ\mathcal{B}^n_p,\ \mathrm{Id}) =$$
$$d_0(\mathcal{B}^n_{f^{-n}p}\circ(\mathcal{A}^{2n}_{f^{-n}p})^{-1}\circ\mathcal{B}^n_p,\ \mathcal{B}^n_{f^{-n}p}\circ(\mathcal{B}^{2n}_{f^{-n}p})^{-1}\circ\mathcal{B}^n_p) =$$
$$d_0(\mathcal{B}^n_{f^{-n}p}\circ(\mathcal{A}^{2n}_{f^{-n}p})^{-1},\ \mathcal{B}^n_{f^{-n}p}\circ(\mathcal{B}^{2n}_{f^{-n}p})^{-1}) \le$$
$$\|D\mathcal{B}^n_{f^{-n}p}\|\, d_0((\mathcal{A}^{2n}_{f^{-n}p})^{-1},\,(\mathcal{B}^{2n}_{f^{-n}p})^{-1}) \le\ c_6\,\sigma^{3n}\lambda^{n\beta} \to 0 \quad \text{as } n\to\infty.$$

Therefore

$$d_0(\mathcal{B}^n_{f^{-n}x}\circ(\mathcal{A}^{2n}_{f^{-n}x})^{-1}\circ\mathcal{B}^n_x,\,\mathrm{Id}) \to 0 \quad \text{as } n\to\infty,$$

and we conclude that $\Psi^s_x = \Psi^u_x$.

3.3. **Extending the conjugacy Ψ to X.**

We denote the homoclinic set of the fixed point p_0 by S, that is,

$$S = W^s(p_0)\cap W^u(p_0).$$

For each $x\in S$, we define

$$\Psi_x = \Psi^s_x = \Psi^u_x = H^{\mathcal{A},s}_{p_0,\,x}\circ H^{\mathcal{B},s}_{x,\,p_0} = H^{\mathcal{A},u}_{p_0,\,x}\circ H^{\mathcal{B},u}_{x,\,p_0}.$$

The homoclinic set S is invariant under f and dense in X [Bo]. The function $\Psi : S\to \mathrm{Diff}^r(\mathcal{M})$ conjugates the cocycles on S. We will show that Ψ is Hölder continuous as function from S to $\mathrm{Diff}^t(\mathcal{M})$. Then it extends to a Hölder continuous function from X to $\mathrm{Diff}^t(\mathcal{M})$ that conjugates the cocycles.

First we establish boundedness of Ψ on S in C^r. We take a sufficiently small $\epsilon>0$, let S' be a finite ϵ-dense subset of S, and let

$$M_{S'} = \max\{\|\Psi_w\|_{C^r} :\ w\in S'\}.$$

Let $x\in S$. Then there is $w\in S'$ with $\mathrm{dist}(x,w)<\epsilon$, and we let $z = W^u_{loc}(x)\cap W^s_{loc}(w)$. Then $z\in S$ and

$$\Psi_x = H^{\mathcal{A},u}_{z,\,x}\circ\Psi_z\circ H^{\mathcal{B},u}_{x,\,z} = H^{\mathcal{A},u}_{z,\,x}\circ H^{\mathcal{A},s}_{w,z}\circ\Psi_w\circ H^{\mathcal{B},s}_{w,\,z}\circ H^{\mathcal{B},u}_{x,\,z}.$$

By property (H3) and (H$\bar{3}$) of the holomomies, there exists a constant M_H such that

$$\|H^{\mathcal{A},u/s}_{x,\,y}\|_{C^r}\le M_H \ \text{ and } \ \|H^{\mathcal{B},u/s}_{x,\,y}\|_{C^r}\le M_H \ \text{ whenever } y\in W^{s/u}_{loc}(x).$$

Applying Lemma 2.3 four times we obtain

$$\|\Psi_x\|_{C^r} \ =\ \|H^{\mathcal{A},u}_{z,\,x}\circ H^{\mathcal{A},s}_{w,z}\circ\Psi_w\circ H^{\mathcal{B},s}_{w,\,z}\circ H^{\mathcal{B},u}_{x,\,z}\|_{C^r} \le$$
$$M_r\,\|H^{\mathcal{A},u}_{z,\,x}\circ H^{\mathcal{A},s}_{w,z}\circ\Psi_w\circ H^{\mathcal{B},s}_{w,\,z}\|_{C^r}\cdot(1+\|H^{\mathcal{B},u}_{x,\,z}\|_{C_r})^r \le$$
$$M_r^2\,\|H^{\mathcal{A},u}_{z,\,x}\circ H^{\mathcal{A},s}_{w,z}\circ\Psi_w\|_{C_r}\cdot(1+\|H^{\mathcal{B},s}_{w,\,z}\|_{C^r})^r\cdot(1+\|H^{\mathcal{B},u}_{x,\,z}\|_{C_r})^r \le\cdots\le$$
$$M_r^4\cdot M_H\cdot(1+M_{s'})^r\cdot(1+M_H)^{3r} =: M_S.$$

Similarly, $\|\Psi_x^{-1}\|_{C^r} \le M_S$.

Let x and y be two sufficiently close points in S. The point $z = W^u_{loc}(x) \cap W^s_{loc}(y)$ is also in S, and $\max\{\operatorname{dist}(x,z), \operatorname{dist}(y,z)\} \le c_7 \operatorname{dist}(x,y)$.

Denoting $H^{\mathcal{A},u}_{z,x} \circ \Psi_z$ by g and applying Lemma 2.5 we obtain

$$\begin{aligned}
&d_{C^t}(\Psi_x, \Psi_z) = d_{C^t}(H^{\mathcal{A},u}_{z,x} \circ \Psi_z \circ H^{\mathcal{B},u}_{x,z},\ \Psi_z) \le \\
&d_{C^t}(H^{\mathcal{A},u}_{z,x} \circ \Psi_z \circ H^{\mathcal{B},u}_{x,z},\ H^{\mathcal{A},u}_{z,x} \circ \Psi_z \circ \mathrm{Id}) + d_{C^t}(H^{\mathcal{A},u}_{z,x} \circ \Psi_z,\ \mathrm{Id} \circ \Psi_z)) \le \\
&M\left(\|g\|_{C^r} + (1 + \|g^{-1}\|_{C^r})^r\right) \cdot d_{C_t}(H^{\mathcal{B},u}_{x,z}, \mathrm{Id})^{r-t} + \\
&\qquad + M(\left(\|\Psi_z\|_{C^r} + (1 + \|\Psi_z^{-1}\|_{C^r}\right)^r) \cdot d_{C_t}(H^{\mathcal{A},u}_{z,x}, \mathrm{Id})^{r-t}.
\end{aligned}$$

Since $\|H^{\mathcal{A},u}_{z,x}\|_{C^r} \le M_H$ and $\|\Psi_x\|_{C^r} \le M_S$, the norm $\|g\|_{C^r}$ is bounded by Lemma 2.3. The "provided that" condition in Lemma 2.5 holds since the quantity (2.11) is uniformly bounded away from 0, and by taking points x and y, and hence x and z, sufficiently close we can make $d_{C_t}(H^{\mathcal{A},u}_{z,x}, \mathrm{Id})$ and $d_{C_t}(H^{\mathcal{B},u}_{z,x}, \mathrm{Id})$ as close to 0 as needed.

Using properties (H3) and (H$\bar{3}$) of the holomomies of $\mathcal{A}$ and $\mathcal{B}$ respectively we obtain

$$d_{C^t}(\Psi_x, \Psi_z) \le M' d(x,z)^{\beta(q-r)(r-t)^2} \le M'' d(x,y)^{\beta(q-r)(r-t)^2}.$$

We conclude that $\Psi : S \to \mathrm{Diff}^t(\mathcal{M})$ is Hölder continuous on S with the exponent $\beta(q-r)(r-t)^2$. Hence extends to a Hölder continuous function $\Psi : X \to \mathrm{Diff}^t(\mathcal{M})$ with the same exponent, and the extension conjugates the cocycles.

3.4. **Removing the fixed point assumption.**

Now suppose that p_0 is a periodic point of f with minimal period $k > 1$, and so p_0 is fixed by f^k.

We consider the iterated cocycles $\mathcal{A}^k$ and $\mathcal{B}^k$ over f^k with $\mathcal{A}^k_{p_0} = \mathcal{B}^k_{p_0}$ and $\Phi_{p_0} = \mathrm{Id}$. The arguments in Sections 3.2 and 3.3 give a Hölder continuous conjugacy $\bar{\Psi} : X \to \mathrm{Diff}^t(\mathcal{M})$ between $\mathcal{A}^k$ and $\mathcal{B}^k$ with $\bar{\Psi}_{p_0} = \mathrm{Id}$. We will show that $\bar{\Psi}$ is also a conjugacy between $\mathcal{A}$ and $\mathcal{B}$ following the approach of [BK16, Sch99].

We consider a point $x \in W^s(p_0) \cap W^u(f^{k-1}(p_0))$, so that $f(x) \in W^u(p_0)$. We will show that

$$\bar{\Psi}_x = \mathcal{A}_x^{-1} \circ \bar{\Psi}_{f(x)} \circ \mathcal{B}_x, \quad \text{that is,} \quad \mathcal{A}_x = \bar{\Psi}_{fx} \circ \mathcal{B}_x \circ \bar{\Psi}_x^{-1}. \tag{3.5}$$

Since such points x are dense in X and all the functions are continuous, it follows that the equation holds for all $x \in \mathcal{M}$.

Now we prove (3.5). Using equations (3.4) we obtain

$$\begin{aligned}
\bar{\Psi}_x &= \bar{\Psi}^s_x = H^{\mathcal{A}^k,s}_{p,x} \circ H^{\mathcal{B}^k,s}_{x,p} = \\
&= \lim_{n\to\infty} (\mathcal{A}^{nk}_x)^{-1} \circ \mathcal{A}^{nk}_p \circ (\mathcal{B}^{nk}_p)^{-1} \circ \mathcal{B}^{nk}_x = \lim_{n\to\infty} (\mathcal{A}^{nk}_x)^{-1} \circ \mathcal{B}^{nk}_x, \quad \text{and} \\
\bar{\Psi}_{fx} &= \bar{\Psi}^u_{fx} = H^{\mathcal{A}^k,u}_{p,fx} \circ H^{\mathcal{B}^k,u}_{fx,p} = \lim_{n\to\infty} \mathcal{A}^{nk}_{f^{-nk}fx} \circ (\mathcal{B}^{nk}_{f^{-nk}fx})^{-1} \\
&= \lim_{n\to\infty} \mathcal{A}^{nk}_{f^{-nk+1}x} \circ (\mathcal{B}^{nk}_{f^{-nk+1}x})^{-1}.
\end{aligned}$$

Then

$$\mathcal{A}_x^{-1} \circ \bar{\Psi}_{fx} \circ \mathcal{B}_x = \lim_{n\to\infty} \mathcal{A}^{nk-1}_{f^{-nk+1}x} \circ (\mathcal{B}^{nk-1}_{f^{-nk+1}x})^{-1}.$$

So we need to show that

$$\lim_{n\to\infty} (\mathcal{A}_x^{nk})^{-1} \circ \mathcal{B}_x^{nk} = \lim_{n\to\infty} \mathcal{A}^{nk-1}_{f^{-nk+1}x} \circ (\mathcal{B}^{nk-1}_{f^{-nk+1}x})^{-1}.$$

Since both limits exist in $\mathrm{Diff}^r(\mathcal{M})$, so does

$$\lim_{n\to\infty} \mathcal{B}^{nk-1}_{f^{-nk+1}x} \circ (\mathcal{A}^{nk-1}_{f^{-nk+1}x})^{-1} \circ (\mathcal{A}_x^{nk})^{-1} \circ \mathcal{B}_x^{nk} = \lim_{n\to\infty} \mathcal{B}^{nk-1}_{f^{-nk+1}x} \circ (\mathcal{A}^{2nk-1}_{f^{-nk+1}x})^{-1} \circ \mathcal{B}_x^{nk},$$

and it suffices to prove that this limit equals the identity. We show that

$$d_0\big(\mathcal{B}^{nk-1}_{f^{-nk+1}x} \circ (\mathcal{A}^{2nk-1}_{f^{-nk+1}x})^{-1} \circ \mathcal{B}_x^{nk},\, \mathrm{Id}\big) \to 0 \ \text{ as } n \to \infty.$$

We recall that $x \in W^s(p_0)$ and $f(x) \in W^u(p_0)$. Hence $f^{-nk+1}(x) = f^{-nk}(f(x))$ is in $W^u(p_0)$ for every $n \in \mathbb{N}$. Therefore by (2.1) there is a constant K_x such that for all $n \in \mathbb{N}$,

$$\mathrm{dist}(f^{nk}x,\, p_0) < \tfrac{1}{2}K_x\lambda^{nk} \quad \text{and} \quad \mathrm{dist}(f^{-nk+1}x,\, p_0) < \tfrac{1}{2}K_x\lambda^{nk}.$$

Let δ' be as in Anosov Closing Lemma 2.1. For all sufficiently large n, we have

$$\delta_n = \mathrm{dist}(f^{nk}x, f^{-nk+1}x) < K_x\lambda^{nk} < \delta',$$

and we can apply the lemma to the the orbit segment $\{f^i x : i = -nk+1, \dots, nk\}$. Thus there exists a periodic point $p_n = f^{2nk-1}(p_n)$ such that

$$\mathrm{dist}(f^i(x),\, f^i(p_n)) \le c'\delta_n < c'K_x\lambda^{nk} \quad \text{for } i = -nk+1,\, \dots, nk.$$

We note that $\mathrm{dist}(p_n, p_0) \le c''\lambda^{nk}$.

The rest of the argument is almost identical to the one in Section 3.2, where we showed that

$$d_0\big(\mathcal{B}^n_{f^{-n}x} \circ (\mathcal{A}^{2n}_{f^{-n}x})^{-1} \circ \mathcal{B}_x^n,\, \mathrm{Id}\big) \to 0 \ \text{ as } n \to \infty.$$

Therefore $\bar{\Psi} : X \to \mathrm{Diff}^t(\mathcal{M})$ is also a conjugacy between $\mathcal{A}$ and $\mathcal{B}$ with $\bar{\Psi}(p_0) = \mathrm{Id}$. This completes the proof of Theorem 1.1.

4. Proofs of Theorems 1.2 and 1.3

The stable and unstable holonomies of the constant cocycle $\mathcal{B}$ are trivial, that is, equal the identity map on $\mathcal{M}$. In Theorem 1.2, condition (2.14) is satisfied by the assumption on $\mathcal{A}$ in (2). In Theorem 1.3, we use the following proposition on cocycles with bounded periodic data.

Proposition 4.1. (From Theorem 1.2(ii) in [S19] and beginning of its proof.)
Let (X, f) be a hyperbolic system, let $\mathcal{M}$ be a compact connected manifold, let $k \in \mathbb{N}$, and let $\mathcal{A}$ be a bounded Diff$^{k+1}(\mathcal{M})$-valued cocycle over f.
If the periodic data set $\{\mathcal{A}^k_p :\ p = f^k p,\ p \in X,\ k \in \mathbb{N}\}$ is bounded in Diff$^1(\mathcal{M})$, then
(i) *The set $\{\mathcal{A}^n_x :\ x \in X,\ k \in \mathbb{Z}\}$ is also bounded in Diff$^1(\mathcal{M})$,*
(ii) *For each $\sigma > 1$ there exists $K > 0$ such that for all $x \in X$,*

$|\mathcal{A}_x^n|_{C^{k+1}} \le K\sigma^{|n|}$ *for all* $n \in \mathbb{Z}$.

By the assumption (1′), the cocycle $\mathcal{B}$ in the Theorem 1.3 has a periodic data set bounded in $\mathrm{Diff}^1(\mathcal{M})$. Since periodic data conjugacy Φ is also bounded in $\mathrm{Diff}^1(\mathcal{M})$ by (2′), it follows that the periodic data of the cocycle $\mathcal{A}$ is bounded in $\mathrm{Diff}^1(\mathcal{M})$. Therefore, $\mathcal{A}$ satisfies the conclusions (i) and (ii) of Proposition 4.1. We take $r = \frac{1}{2}(t+q)$. Then $\mathcal{A}$ has stable holonomies as in Proposition 2.6, as well as the unstable holonomies.

Thus in Theorems 1.3 and 1.2, the cocycle $\mathcal{A}$ has stable and unstable holonomies in $\mathrm{Diff}^r(\mathcal{M})$ as in Proposition 2.6. Since the holonomies of $\mathcal{B}$ are trivial, we will write just $H^s_{x,y}$ and $H^u_{x,y}$ for the holonomies $H^{\mathcal{A},s}_{x,y}$ and $H^{\mathcal{A},u}_{x,y}$ of $\mathcal{A}$.

4.1. **Construction of a conjugacy Ψ on the homoclinic set of a fixed point.** Let p_0 be a fixed point of f. As in the proof of Theorem 1.1 we conjugate the cocycle $\mathcal{B}$ by $\Phi(p_0)$ and obtain another constant cocycle. So we assume that

$$\mathcal{A}_{p_0} = \mathcal{B}_{p_0} = B \quad \text{and} \quad \Phi_{p_0} = \mathrm{Id}.$$

We will construct a Hölder continuous conjugacy $\Psi : X \to \mathrm{Diff}^t(\mathcal{M})$ between $\mathcal{A}$ and the adjusted cocycle with the exponent $\beta(q-r)(r-t)$. Then $\Psi = \tilde{\Psi} \circ \Phi_{p_0}$ will be a conjugacy between $\mathcal{A}$ and and the original $\mathcal{B}$ with values in $\mathrm{Diff}^t(\mathcal{M})$, Hölder continuous with exponent $\beta(q-r)(r-t)(q-t)$.

First we define conjugacies Ψ^s and Ψ^u on the stable and unstable leaves of the fixed point q. We set

$$\begin{aligned} &\Psi^s_{p_0} = \Psi^u_{p_0} = \Phi_{p_0} = \mathrm{Id}, \\ &\Psi^s_x = H^s_{p_0,x} \quad \text{for } x \in W^s(p_0), \\ &\Psi^u_x = H^u_{p_0,x} \quad \text{for } x \in W^u(p_0). \end{aligned}$$

By the property (H2) of the holonomies we have $\mathcal{A}_x = H^s_{fx,fy} \circ \mathcal{A}_y \circ H^s_{x,y}$. Using this as well as $p_0 = f(p_0)$ and $\mathcal{A}_{p_0} = B$, we obtain

$$\Psi^s_{fx} \circ B \circ (\Psi^s_x)^{-1} = H^{s/u}_{p_0,fx} \circ B \circ (H^{s/u}_{f(p_0),\,x})^{-1} = H^s_{p_0,f(x)} \circ \mathcal{A}_{p_0} \circ H^s_{x,\,p_0} = \mathcal{A}_x.$$

So Ψ^s is a conjugacy between $\mathcal{A}_x$ and B on $W^s(p_0)$, and similarly Ψ^u is a conjugacy on $W^u(p_0)$.

Let x be a *homoclinic point* for q, that is, $x \in W^s(p_0) \cap W^u(p_0)$. We will show that

$$\Psi^s_x = \Psi^u_x, \quad \text{that is,} \quad (\Psi^s_x)^{-1} \circ \Psi^u_x = H^s_{x,p_0} \circ H^u_{x,p_0} = \mathrm{Id}.$$

In Theorem 1.3, since the diffeomorphisms $\{B^m : m \in \mathbb{N}\}$ have uniformly bounded first derivatives and $\mathcal{M}$ is compact, the sequence $\{B^m : m \in \mathbb{N}\}$ is uniformly bounded and equicontinuous. Hence it has a subsequence $\{B^{m_\ell}\}$ that converges uniformly on $\mathcal{M}$ to a continuous map $\tilde{B}$. Then we choose a subsequence $\{m_{\ell_j}\}$ of $\{m_\ell\}$ such that $\{B^{-m_{\ell_j}}\}$ converges uniformly on $\mathcal{M}$ to a continuous map $\bar{B}$. In Theorem 1.2, such a sequence is obtained using pre-compactness of $\{B^m : m \in \mathbb{Z}\}$ in $C^0(\mathcal{M})$. We have $\tilde{B} \circ \bar{B} = \bar{B} \circ \tilde{B} = \mathrm{Id}$, and so $\tilde{B}$ is a homeomorphism. Thus there is a sequence $\{m_{\ell_j}\}$ such

that $\{B^{m_{\ell_j}}\}$ converges to a homeomorphism $\tilde{B}$ in Homeo($\mathcal{M}$). It follows that there is an increasing sequence $\{n_k\}$ of the form $n_k = m_\ell - m_{\ell'}$ such that $\{B^{n_k}\}$ converges to the identity map on $\mathcal{M}$ in Homeo($\mathcal{M}$).

Since $\mathcal{A}_{p_0} = B$, we also have

$$\mathcal{A}_{p_0}^{n_k} \to \mathrm{Id} \quad \text{in Homeo}(\mathcal{M}).$$

For this sequence $\{n_k\}$ the holonomies of $\mathcal{A}$ can be written as

$$H^s_{x,p_0} = \lim_{n\to\infty} (\mathcal{A}^n_{p_0})^{-1} \circ \mathcal{A}^n_x = \lim_{k\to\infty} (\mathcal{A}^{n_k}_{p_0})^{-1} \circ \mathcal{A}^{n_k}_x) = \lim_{k\to\infty} \mathcal{A}^{n_k}_x,$$

where convergence is in Homeo($\mathcal{M}$). Similarly, since $p_0 = f^{-n_k}(p_0)$, we have

$$H^u_{p_0,x} = \lim_{k\to\infty} (\mathcal{A}^{-n_k}_x)^{-1} \circ \mathcal{A}^{-n_k}_{p_0} = \lim_{k\to\infty} \mathcal{A}^{n_k}_{f^{-n_k}(x)} \circ \big(\mathcal{A}^{n_k}_{f^{-n_k}(p_0)}\big)^{-1} = \lim_{k\to\infty} \mathcal{A}^{n_k}_{f^{-n_k}(x)}.$$

Therefore, with convergence in Homeo($\mathcal{M}$), we have

$$(\Psi^s_x)^{-1} \circ \Psi^u_x = H^s_{x,p_0} \circ H^u_{p_0,x} = \lim_{k\to\infty} \mathcal{A}^{n_k}_x \circ \mathcal{A}^{n_k}_{f^{-n_k}(x)} = \lim_{k\to\infty} \mathcal{A}^{2n_k}_{f^{-n_k}(x)}. \tag{4.1}$$

We will show that $\mathcal{A}^{2n_k}_{f^{-n_k}(x)} \to \mathrm{Id}$.

Let δ' be as in the Anosov Closing Lemma 2.1. We consider an orbit segment $\{f^{-n_k}x, \dots, x, \dots f^{n_k}(x)\}$. For all sufficiently large k we have

$$\delta_k = \mathrm{dist}(f^{-n_k}x, f^{n_k}(x)) \le K_x \lambda^{n_k} < \delta'.$$

Hence there exists a periodic point $p_k = f^{2n_k}(p_k)$ whose orbit $\{p_k, \dots, f^{2n_k}(p_k)\}$ is close to the orbit segment above, and in particular

$$\mathrm{dist}(f^{-n_k}x, p_k) \le c'\delta_k \quad \text{and} \quad \mathrm{dist}((f^{n_k}x), p_k) \le c'\delta_k.$$

Now we show that $\mathcal{A}^{2n_k}_{f^{-n_k}(x)}$ is close to $\mathcal{A}^{2n_k}_{p_k}$. By the local product structure there exits a unique point

$$y = W^s_{loc}(f^{-n_k}x) \cap W^u_{loc}(p_k).$$

Then $\mathrm{dist}(f^{-n_k}x, y) \le c_1\delta_k$, and the orbit segment $\{y, \dots, f^{2n_k}(x)\}$ is close to both orbit segments $\{f^{-n_k}x, \dots, f^{n_k}(x)\}$ and $\{p_k, \dots, f^{2n_k}(p_k) = p_k\}$. It follows that

$$f^{2n_k}y = W^s_{loc}(f^{n_k}x) \cap W^u_{loc}(p_k), \text{ and so } \mathrm{dist}(f^{2n_k}y, p_k) \le c_1\delta_k.$$

Then applying (2.16) with d_{C^0} to $f^{-n_k}x$ and y, and to $f^{2n_k}y$ and $f^{2n_k}p_k = p_k$ we obtain

$$d_{C^0}((\mathcal{A}^{2n_k}_y)^{-1} \circ \mathcal{A}^{2n_k}_{f^{-n_k}(x)}, \mathrm{Id}) \le c\,\mathrm{dist}(y, f^{-n_k}(x))^{\beta(q-r)} \le c(c_1\delta_k)^{\beta(q-r)} \quad \text{and}$$
$$d_{C^0}(\mathcal{A}^{2n_k}_{p_k} \circ (\mathcal{A}^{2n_k}_y)^{-1}, \mathrm{Id}) \le c(c_1\delta_k)^{\beta(q-r)}.$$

Now we show that $d_0(\mathcal{A}^{2n_k}_{f^{-n_k}(x)}, \mathcal{A}^{2n_k}_{p_k}) \to 0$. We have

$$d_0(\mathcal{A}^{2n_k}_{f^{-n_k}(x)}, \mathcal{A}^{2n_k}_{p_k}) \le d_0(\mathcal{A}^{2n_k}_{f^{-n_k}(x)}, \mathcal{A}^{2n_k}_y) + d_{C^0}(\mathcal{A}^{2n_k}_y, \mathcal{A}^{2n_k}_{p_k}) =$$
$$d_0(\mathcal{A}^{2n_k}_y \circ \big(\mathcal{A}^{2n_k}_y\big)^{-1} \circ \mathcal{A}^{2n_k}_{f^{-n_k}(x)}, \mathcal{A}^{2n_k}_y\big) + d_0\big(\mathcal{A}^{2n_k}_y, \mathcal{A}^{2n_k}_{p_k} \circ (\mathcal{A}^{2n_k}_y)^{-1} \circ \mathcal{A}^{2n_k}_y\big) =$$
$$d_0(\mathcal{A}^{2n_k}_y \circ \big(\mathcal{A}^{2n_k}_y\big)^{-1} \circ \mathcal{A}^{2n_k}_{f^{-n_k}(x)}, \mathcal{A}^{2n_k}_y\big) + d_0\big(\mathrm{Id}, \mathcal{A}^{2n_k}_{p_k} \circ (\mathcal{A}^{2n_k}_y)^{-1}\big),$$

and the last term is at most $c(c_1\delta_k)^{\beta(q-r)} \to 0$ as $k \to \infty$.

In Theorem 1.3, the iterates of $\mathcal{A}$ have uniformly bounded first derivatives, and so

$$d_0(\mathcal{A}_y^{2n_k} \circ \big(\mathcal{A}_y^{2n_k}\big)^{-1} \circ \mathcal{A}_{f^{-n_k}(x)}^{2n_k},\, \mathcal{A}_y^{2n_k}\big) \le c_2 \cdot d_0\big((\mathcal{A}_y^{2n_k})^{-1} \circ \mathcal{A}_{f^{-n_k}(x)}^{2n_k},\, \mathrm{Id}\big) \le c_4 \delta_k^{\beta(q-r)}.$$

In Theorem 1.2, we use the assumption (2). The condition $\sigma^{2(1+r)}\lambda^{\beta(q-r)} < 1$ implies $\sigma^2\lambda^{\beta(q-r)} < 1$, and we estimate

$$\begin{aligned} d_0(\mathcal{A}_y^{2n_k} \circ \big(\mathcal{A}_y^{2n_k}\big)^{-1} \circ \mathcal{A}_{f^{-n_k}(x)}^{2n_k},\, \mathcal{A}_y^{2n_k}\big) &\le \|D\mathcal{A}_y^{2n_k}\| \cdot d_0(\big(\mathcal{A}_y^{2n_k}\big)^{-1} \circ \mathcal{A}_{f^{-n_k}(x)}^{2n_k},\, \mathrm{Id}\big) \\ \le K\sigma^{2n_k} \cdot c(c_1\delta_k)^{\beta(q-r)} &\le K\sigma^{2n_k} \cdot c_4\lambda^{n_k\beta(q-r)} \le c_5(\sigma^2\lambda^{\beta(q-r)})^{n_k} \to 0 \text{ as } k \to \infty. \end{aligned}$$

Thus in both settings we have

$$d_0(\mathcal{A}_{f^{-n_k}(x)}^{2n_k},\, \mathcal{A}_{p_k}^{2n_k}) \to 0 \quad \text{as } k \to \infty.$$

We recall that $\mathcal{A}_{p_k}^{2n_k} = \Phi(p_k)^{-1} \circ B^{2n_k} \circ \Phi(p_k)$, and that $B^{2n_k} \to \mathrm{Id}$ in C^0 since $B^{n_k} \to \mathrm{Id}$ in C^0. We have

$$d_0(\mathcal{A}_{p_k}^{2n_k},\, \mathrm{Id}) \,=\, d_0(\Phi(p_k)^{-1} \circ B^{2n_k} \circ \Phi(p_k),\, \mathrm{Id}) \,=\, d_0(\Phi(p_k)^{-1} \circ B^{2n_k},\, \Phi(p_k)^{-1}).$$

In Theorem 1.3, since the set $\{\Phi(p)\}$ is bounded in $\mathrm{Diff}^1(\mathcal{M})$, this is at most $c_6\, d_0(B^{2n_k}, \mathrm{Id})$, which tends to zero. In Theorem 1.2, the family of maps $\{\Phi(p)^{-1}\}$ is pre-compact in C^0 and hence equicontinuous, and it follows that

$$d_0(\Phi(p_k)^{-1} \circ B^{2n_k},\, \Phi(p_k)^{-1}) \to 0.$$

Thus

$$d_0(\mathcal{A}_{p_k}^{2n_k},\, \mathrm{Id}) \to 0 \quad \text{and hence} \quad \mathcal{A}_{f^{-n_k}(x)}^{2n_k} \to \mathrm{Id},$$

and using (4.1) we conclude that

$$(\Psi_x^s)^{-1} \circ \Psi_x^u \,=\, \lim_{k\to\infty} \mathcal{A}_{f^{-n_k}(x)}^{2n_k} = \mathrm{Id}, \quad \text{that is,} \quad \Psi_x^s = \Psi_x^u.$$

We denote the set of homoclinic points for the fixed point p_0 by S. For each $x \in S$ we define

$$\Psi_x = \Psi_x^s = \Psi_x^u,$$

which gives a conjugacy between $\mathcal{A}$ and $\mathcal{B}$ on S. Since $\Psi_x = H_{p_0,x}^s$, and the holonomies are in $\mathrm{Diff}^r(\mathcal{M})$, we have $\Psi_x \in \mathrm{Diff}^r(\mathcal{M})$.

Extending the conjugacy Ψ to X is done as in the proof of Theorem 1.1, but the estimates are simpler since the holonomies of $\mathcal{B}$ are trivial.

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