COHOMOLOGY OF FIBER BUNCHED COCYCLES OVER HYPERBOLIC SYSTEMS

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Abstract. We consider Hölder continuous fiber bunched $GL(d,\mathbb{R})$-valued cocycles over an Anosov diffeomorphism. We show that two such cocycles are Hölder continuously cohomologous if they have equal periodic data, and prove a result for cocycles with conjugate periodic data. We obtain a corollary for cohomology between *any* constant cocycle and its small perturbation. The fiber bunching condition means that non-conformality of the cocycle is dominated by the expansion and contraction in the base. We show that this condition can be established based on the periodic data. Some important examples of cocycles come from the differential of the diffeomorphism and its restrictions to invariant sub-bundles. We discuss an application of our results to the question when an Anosov diffeomorphism is smoothly conjugate to a C^1-small perturbation. We also establish Hölder continuity of a measurable conjugacy between a fiber bunched cocycle and a uniformly quasiconformal one. Our main results also hold for cocycles with values in a closed subgroup of $GL(d,\mathbb{R})$, for cocycles over hyperbolic sets and shifts of finite type, and for linear cocycles on a non-trivial vector bundle.

1. Inroduction

Cocycles and their cohomology arise naturally in the theory of group actions and play an important role in dynamics. In this paper we study cohomology of Hölder continuous group-valued cocycles over hyperbolic dynamical systems. Our motivation comes in part from questions in local and global rigidity for hyperbolic systems and actions, where the derivative and the Jacobian provide important examples of cocycles. We state our results for the case of an Anosov diffeomorphism, but they also hold for cocycles over hyperbolic sets and symbolic dynamical systems.

Definition 1.1. *Let f be a diffeomorphism of a compact manifold $\mathcal{M}$ and let A be a Hölder continuous function from $\mathcal{M}$ to a metric group G. The G-valued cocycle over f generated by A is the map $\mathcal{A}:\mathcal{M}\times\mathbb{Z}\to G$ defined by*

$$\mathcal{A}(x,0)=\mathcal{A}_x^0=e_G,\quad \mathcal{A}(x,n)=\mathcal{A}_x^n=A(f^{n-1}x)\circ\cdots\circ A(x) \quad \textit{and}$$
$$\mathcal{A}(x,-n)=\mathcal{A}_x^{-n}=(\mathcal{A}_{f^{-n}x}^n)^{-1}=(A(f^{-n}x))^{-1}\circ\cdots\circ(A(f^{-1}x))^{-1},\quad n\in\mathbb{N}.$$

If the tangent bundle of $\mathcal{M}$ is trivial, i.e. $T\mathcal{M}=\mathcal{M}\times\mathbb{R}^d$, then the differential Df can be viewed as a $GL(d,\mathbb{R})$-valued cocycle: $A_x=Df_x$ and $\mathcal{A}_x^n=Df_x^n$. More generally,

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one can consider restrictions of Df to invariant sub-bundles of $T\mathcal{M}$, for example stable and unstable. Typically, these sub-bundles are only Hölder continuous, and hence so are the corresponding cocycles. On the other hand, Hölder regularity is necessary to develop a meaningful theory for cocycles over hyperbolic systems, even in the simplest case of $G = \mathbb{R}$.

Definition 1.2. *Cocycles $\mathcal{A}$ and $\mathcal{B}$ are (measurably, continuously)* cohomologous *if there exists a (measurable, continuous) function $C : \mathcal{M} \to G$ such that*

$$\mathcal{A}_x^n = C(f^n x) \circ \mathcal{B}_x^n \circ C(x)^{-1} \quad \text{for all } n \in \mathbb{Z} \text{ and } x \in \mathcal{M}, \tag{1.1}$$

equivalently, $\mathcal{A}_x = C(fx) \circ \mathcal{B}_x \circ C(x)^{-1}$ for all $x \in \mathcal{M}$.

We refer to C as a conjugacy *between $\mathcal{A}$ and $\mathcal{B}$. It is also called a transfer map.*

Hölder continuous cocycles over hyperbolic systems have been extensively studied starting with the seminal work of A. Livšic [Liv71, Liv72]. The research has been focused on obtaining sufficient conditions for cohomology in terms of the periodic data and on studying the regularity of the conjugacy C, see [KtN] for an overview.

Definition 1.3. *Cocycles $\mathcal{A}$ and $\mathcal{B}$ have* conjugate periodic data *if for every periodic point $p = f^n(p)$ in $\mathcal{M}$ there exists $C(p) \in G$ such that*

$$\mathcal{A}_p^n = C(p) \circ \mathcal{B}_p^n \circ C(p)^{-1}. \tag{1.2}$$

Clearly, having conjugate periodic data is a necessary condition for continuous cohomology of two cocycles, and it is natural to ask whether it is also sufficient. If G is an abelian group, the problem reduces to the case when $\mathcal{B}$ is the identity cocycle, i.e. $\mathcal{B}_x = e_G$, and the periodic assumption is simply $\mathcal{A}_p^n = e_G$. The positive answer for this case was given by A. Livšic [Liv71]. Even for non-abelian G, the case of $\mathcal{B} = e_G$ has been studied most and by now is relatively well understood, see for example [Liv72, NT95, PW01, LW10, K11].

For non-abelian G, however, the general problem does not reduce to the special case $\mathcal{B} = e_G$ and is much more difficult. There are very few results for non-abelian groups, and almost none beyond the essentially compact case. Even when $C(p)$ is bounded the answer is negative in general [S13]. If $C(p)$ is Hölder, conjugating $\mathcal{B}$ by the extension of C reduces the problem to the case of equal periodic data, i.e. $\mathcal{A}_p^n = \mathcal{B}_p^n$. Positive results for equal periodic data, as well as some results for conjugate data, were established by W. Parry [Pa99] for compact G and, somewhat more generally, by K. Schmidt [Sch99] for cocycles with "bounded distortion". First results outside this setting were obtained in [S13] for certain types of $GL(2, \mathbb{R})$-valued cocycles.

In this paper we consider fiber bunched cocycles with values in $GL(d, \mathbb{R})$ or its closed subgroup. We establish Hölder cohomology for cocycles with equal periodic data and prove a result for cocycles with conjugate periodic data under a mild regularity assumption on $C(p)$. The fiber bunching condition (2.2) means that non-conformality of the cocycle is, in a sense, dominated by expansion and contraction in the base.

In particular, conformal and uniformly quasiconformal cocycles satisfy this condition. Fiber bunching and similar assumptions ensure convergence of certain iterates of the cocycle and play a crucial role in the non-commutative case. We show that fiber bunching can be obtained from the periodic data, and hence we assume it for only one of the cocycles. We obtain a corollary for perturbations of *any* constant cocycle, not necessarily fiber bunched.

We also consider a related question whether a measurable solution C of (1.1) is necessarily continuous. Even the case of $\mathcal{B} = e_G$ remains open in full generality, but positive answers were obtained under additional assumptions [Liv72, GSp97, NP99, PW01]. The case of two arbitrary cocycles with values in a compact group was resolved affirmatively by W. Parry and M. Pollicott [PaP97], and by K. Schmidt [Sch99] for cocycles with "bounded distortion". Positive results for certain types of $GL(2,\mathbb{R})$-valued cocycles were obtained in [S13]. On the other hand, examples of $GL(2,\mathbb{R})$-valued cocycles which are measurably but not continuously cohomologous were constructed in [PW01], moreover both cocycles can be made arbitrarily close to the identity. This shows that fiber bunching of the cocycles does not ensure continuity of C. In this paper we establish Hölder continuity of a measurable conjugacy under a stronger assumption that one cocycle is fiber bunched and the other one is uniformly quasiconformal. For smooth cocycles, higher regularity of the conjugacy then follows from [NT98].

We state the results on cohomology of cocycles in Section 2 and and give the proofs in Section 4. We describe other settings for our results in Section 3. In Section 5 we discuss an application to the question when an Anosov diffeomorphism is smoothly conjugate to a C^1-small perturbation.

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2. Statement of results on cohomology of cocycles

Anosov diffeomorphisms. Let $\mathcal{M}$ be a compact connected Riemannian manifold. We recall that a diffeomorphism f of $\mathcal{M}$ is called *Anosov* if there exist a splitting of the tangent bundle $T\mathcal{M}$ into a direct sum of two Df-invariant continuous subbundles E^s and E^u, a Riemannian metric on $\mathcal{M}$, and continuous functions ν and $\hat{\nu}$ such that

$$\|Df(\mathbf{v}^s)\| < \nu(x) < 1 < \hat{\nu}(x) < \|Df(\mathbf{v}^u)\| \tag{2.1}$$

for any $x \in \mathcal{M}$ and unit vectors $\mathbf{v}^s \in E^s(x)$ and $\mathbf{v}^u \in E^u(x)$. The distributions E^s and E^u are called stable and unstable. They are tangent to the stable and unstable foliations W^s and W^u respectively (see, for example [KtH]). A diffeomorphism is said to be *transitive* if there is a point x in $\mathcal{M}$ with dense orbit. All known examples of Anosov diffeomorphisms have this property.

Standing assumptions. In this paper,
f is a C^2 transitive Anosov diffeomorphism of a compact connected manifold $\mathcal{M}$,
$\mathcal{A}$ and $\mathcal{B}$ are β-Hölder continuous $GL(d,\mathbb{R})$-valued cocycles over f.

We denote by $\|A\|$ the operator norm of the matrix A and we use the following distance on $GL(d,\mathbb{R})$: $d(A,B) = \|A-B\| + \|A^{-1} - B^{-1}\|$.

A $GL(d,\mathbb{R})$-valued cocycle $\mathcal{A}$ is β-Hölder continuous if there exist constant c such that $d(\mathcal{A}_x, \mathcal{A}_y) \le c \,\mathrm{dist}(x,y)^\beta$ for all $x, y \in \mathcal{M}$.

Definition 2.1. *A β-Hölder continuous cocycle $\mathcal{A}$ over an Anosov diffeomorphism f is* fiber bunched *if there exist numbers $\theta < 1$ and L such that for all $x \in \mathcal{M}$ and $n \in \mathbb{N}$,*

$$\|\mathcal{A}_x^n\| \cdot \|(\mathcal{A}_x^n)^{-1}\| \cdot (\nu_x^n)^\beta < L\,\theta^n \quad and \quad \|\mathcal{A}_x^{-n}\| \cdot \|(\mathcal{A}_x^{-n})^{-1}\| \cdot (\hat{\nu}_x^{-n})^\beta < L\,\theta^n, \tag{2.2}$$

where $\nu_x^n = \nu(f^{n-1}x) \cdots \nu(x)$ *and* $\hat{\nu}_x^{-n} = (\hat{\nu}(f^{-n}x))^{-1} \cdots (\hat{\nu}(f^{-1}x))^{-1}$.

First we establish Hölder cohomology for cocycles with equal periodic data.

Theorem 2.2. *Suppose that a cocycle $\mathcal{A}$ is fiber bunched and a cocycle $\mathcal{B}$ has the same periodic data, i.e. $\mathcal{B}_p^n = \mathcal{A}_p^n$ whenever $f^n(p) = p$. Then $\mathcal{A}$ and $\mathcal{B}$ are β-Hölder continuously cohomologous. Moreover, if $\mathcal{A}$ and $\mathcal{B}$ take values in a closed subgroup of $GL(d,\mathbb{R})$, then a β-Hölder continuous conjugacy between them can be chosen in the same subgroup.*

In this theorem we assume fiber bunching only for $\mathcal{A}$, as for $\mathcal{B}$ it follows from the proposition below. We give a necessary and sufficient condition for a cocycle to be fiber bunched in terms of its periodic data in Corollary 4.2.

Proposition 2.3. *Suppose that a cocycle $\mathcal{A}$ is fiber bunched and $\mathcal{B}$ has conjugate periodic data. Then $\mathcal{B}$ is also fiber bunched.*

Now we consider the question whether conjugacy of the periodic data for two cocycles implies cohomology. The case of Hölder congugacy of the periodic data easily reduces to the case of equality. Indeed, one can extend the Hölder continuous function $C(p)$ to $\mathcal{M}$ and consider the cocycle $\tilde{\mathcal{B}}_x = C(fx) \circ \mathcal{B}_x \circ C(x)$ so that $\mathcal{A}$ and $\tilde{\mathcal{B}}$ have equal periodic data. By Theorem 2.2 the cocycles $\mathcal{A}$ and $\tilde{\mathcal{B}}$ are Hölder cohomologous, and hence so are $\mathcal{A}$ and $\mathcal{B}$.

On the other hand, Example 2.7 in [S13] shows that boundedness assumption for the conjugacy is too weak: arbitrarily close to the identity, there exist smooth $GL(2,\mathbb{R})$-valued cocycles that have conjugate periodic data with $C(p)$ uniformly bounded, but are not even measurably cohomologous.

In the next theorem we assume that the diffeomorphism f has a fixed point. It is an open question whether every Anosov diffeomorphism satisfies this assumption. We obtain Hölder cohomology of the cocycles if $C(p)$ is Hölder continuous at a fixed point. If we assume that $C(p)$ is Hölder continuous at a periodic point $p = f^N p$, then the theorem yields Hölder cohomology of the iterates $\mathcal{A}^N$ and $\mathcal{B}^N$ over f^N.

Theorem 2.4. *Suppose that $\mathcal{A}$ is fiber bunched and $\mathcal{B}$ has conjugate periodic data. In addition, suppose that f has a fixed point p_0 and the conjugacy $C(p)$ is β-Hölder continuous at p_0, i.e. $d(C(p), C(p_0)) \le c\, dist(p, p_0)^\beta$ for every periodic point p.*

Then $C(p_0)$ extends to a unique β-Hölder continuous conjugacy C between $\mathcal{A}$ and $\mathcal{B}$. Moreover, if $\mathcal{A}$, $\mathcal{B}$, and $C(p_0)$ take values in a closed subgroup G_0 of $GL(d,\mathbb{R})$, then $C(x) \in G_0$ for all x.

The corollary below gives a similar result for a constant cocycle and its perturbation without the fiber bunching assumption. The proof is outlined in the end of Section 5.

Corollary 2.5. *Suppose that $\mathcal{A}$ is a constant cocycle, and $\mathcal{B}$ is sufficiently close to $\mathcal{A}$ and has conjugate periodic data. In addition, suppose that f has a fixed point p_0 and $C(p)$ is Hölder continuous at p_0. Then $\mathcal{A}$ and $\mathcal{B}$ are Hölder continuously cohomologous.*

Next we consider the question whether a measurable conjugacy between two fiber bunched cocycles is continuous. An example in [PW01] demonstrates that the answer is negative in general: arbitrarily close to the identity, there exist smooth $GL(d,\mathbb{R})$-valued cocycles that are are measurably, but not continuously cohomologous. Thus we make a stronger assumption that one of the cocycles is uniformly quasiconformal.

Definition 2.6. *A cocycle $\mathcal{B}$ is called* uniformly quasiconformal *if the quasiconformal distortion $K_{\mathcal{B}}(x,n) = \|\mathcal{B}_x^n\| \cdot \|(\mathcal{B}_x^n)^{-1}\|$ is uniformly bounded for all $x \in \mathcal{M}$ and $n \in \mathbb{Z}$. If $K_{\mathcal{B}}(x,n) = 1$ for all x and n, the cocycle is said to be* conformal.

Theorem 2.7. *Suppose that $\mathcal{A}$ is fiber bunched and $\mathcal{B}$ is uniformly quasiconformal. Let μ be an ergodic invariant measure with full support and local product structure.*

Then any μ-measurable conjugacy between $\mathcal{A}$ and $\mathcal{B}$ is β-Hölder continuous, i.e. it coincides with a β-Hölder continuous conjugacy on a set of full measure.

A measure has local product structure if it is locally equivalent to the product of its conditional measures on the local stable and unstable manifolds. Examples of ergodic measures with full support and local product structure include the measure of maximal entropy, more generally Gibbs (equilibrium) measures of Hölder continuous potentials, and the invariant volume if it exists [PW01].

3. Other settings

Other systems in the base. Our results hold and the proofs apply without significant modifications to $GL(d,\mathbb{R})$ -valued cocycles over mixing locally maximal hyperbolic sets and over mixing shifts of finite type. Mixing holds automatically for transitive Anosov diffeomorphisms of connected manifolds. We briefly describe the other two settings.

1. Cocycles over hyperbolic sets. (See [KtH] for more details.) Let f be a diffeomorphism of a manifold $\mathcal{M}$. A compact f-invariant set $\Lambda \subset \mathcal{M}$ is called *hyperbolic* if there exist a continuous Df-invariant splitting $T_\Lambda \mathcal{M} = E^s \oplus E^u$, and a Riemannian metric and continuous functions ν, $\hat{\nu}$ on an open set $U \supset \Lambda$ such that (2.1) holds for all $x \in \Lambda$. A β-Hölder cocycle over the map $f|_\Lambda$ is fiber bunched if (2.2) holds on Λ.

The set Λ is called *locally maximal* if $\Lambda = \bigcap_{n\in\mathbb{Z}} f^{-n}(U)$ for some open set $U \supset \Lambda$. The map $f|_\Lambda$ is called *topologically mixing* if for any two open non-empty subsets U, V of Λ there is $N \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ for all $n \geq N$.

2. Cocycles over shifts of finite type. Let Q be $k \times k$ matrix with entries from $\{0,1\}$ such that all entries of Q^N are positive for some N. Let

$$\Sigma = \{\, x = (x_n)_{n\in\mathbb{Z}} \;\mid\; 1 \le x_n \le k \;\text{ and }\; Q_{x_n,x_{n+1}} = 1 \;\text{ for every } n \in \mathbb{Z}\,\}.$$

The shift map $\sigma : \Sigma \to \Sigma$ is defined by $(\sigma(x))_n = x_{n+1}$. The system (Σ, σ) is called a mixing *shift of finite type*. Σ has a natural family of metrics d_α, $\alpha \in (0,1)$, defined by

$$d_\alpha(x,y) = \alpha^{n(x,y)}, \quad \text{where} \quad n(x,y) = \min\{\, |i| \;\mid\; x_i \ne y_i\}.$$

The following sets play the role of the local stable and unstable manifolds of x:

$$W^s_{loc}(x) = \{\, y \mid\; x_i = y_i, \;\; i \ge 0\,\}, \quad W^u_{loc}(x) = \{\, y \mid\; x_i = y_i, \;\; i \le 0\,\},$$

indeed for $n \in \mathbb{N}$,

$$d_\alpha(\sigma^n(x), \sigma^n(y)) = \alpha^n\, d_\alpha(x,y) \quad \text{for } y \in W^s_{loc}(x),$$
$$d_\alpha(\sigma^{-n}(x), \sigma^{-n}(y)) = \alpha^n\, d_\alpha(x,y) \quad \text{for } y \in W^u_{loc}(x).$$

Hence the main distance estimate (4.3) in our proofs holds with $\nu = \alpha$ and $\hat\nu = 1/\alpha$. A β-Hölder cocycle $\mathcal{A}$ over $(\Sigma, \sigma, d_\alpha)$ is fiber bunched if there are $\theta < 1$ and L such that

$$\|\mathcal{A}^n_x\| \cdot \|(\mathcal{A}^n_x)^{-1}\| \cdot \alpha^{\beta|n|} < L\,\theta^{|n|} \quad \text{for all } n \in \mathbb{Z}.$$

Linear cocycles over an Anosov diffeomorphism. A $GL(d,\mathbb{R})$-valued cocycle over f can be viewed as an automorphism of the trivial vector bundle $\mathcal{E} = \mathcal{M} \times \mathbb{R}^d$. More generally, we can consider *linear cocycles* over f, i.e. automorphisms of a d-dimensional vector bundle $\mathcal{E}$ over $\mathcal{M}$ covering f, see [KS13] for details of this setting including Hölder regularity. The results (except for statements about subgroups) and the proofs extend directly to this context.

4. PROOFS

4.1. **Fiber bunching and periodic data.**

In this section we prove Proposition 2.3 and then we formulate the fiber bunching condition in terms of the periodic data.

Proof of Proposition 2.3. The proof relies on the following result on subadditive sequences. Let f be a homeomorphism of a compact metric space X. A sequence of continuous functions $a_n : X \to \mathbb{R}$ is called *subadditive* if

$$a_{n+k}(x) \le a_k(x) + a_n(f^k x) \quad \text{for all } x \in X \text{ and } n,k \in \mathbb{N}.$$

Let μ be an f-invariant Borel probability measure on X and let $a_n(\mu) = \int_X a_n d\mu$. Then $a_{n+k}(\mu) \le a_n(\mu) + a_k(\mu)$, i.e. the sequence of real numbers $\{a_n(\mu)\}$ is subadditive. It is well known that for such a sequence the following limit exists:

$$\chi(a,\mu) := \lim_{n\to\infty} \frac{a_n(\mu)}{n} = \inf_{n\in\mathbb{N}} \frac{a_n(\mu)}{n}.$$

Also, by the Subadditive Ergodic Theorem, if the measure μ is ergodic then

$$\lim_{n\to\infty} \frac{a_n(x)}{n} = \chi(a,\mu) \quad \text{for } \mu\text{-almost all } x \in X.$$

Lemma 4.1. [KS13, Proposition 4.9] *Let f be a homeomorphism of a compact metric space X and $a_n : X \to \mathbb{R}$ be a subadditive sequence of continuous functions.*

If $\chi(a, \mu) < 0$ for every ergodic invariant Borel probability measure μ for f, then there exists N such that $a_N(x) < 0$ for all $x \in X$.

We will apply this result to the sequence of functions

$$a_n(x) = \log\left(\|\mathcal{B}_x^n\| \cdot \|(\mathcal{B}_x^n)^{-1}\| \cdot (\nu_x^n)^\beta\right).$$

It is easy to verify that this sequence is subadditive. To show that it satisfies the assumption of Proposition 4.1, we consider Lyapunov exponents of cocycles.

Let μ be an ergodic f-invariant measure, and let $\lambda_+(\mathcal{B}, \mu)$ and $\lambda_-(\mathcal{B}, \mu)$ be the largest and smallest Lyapunov exponents of $\mathcal{B}$ with respect to μ. We recall that

$$\lambda_+(\mathcal{B}, \mu) = \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{B}_x^n\| \quad \text{and} \quad \lambda_-(\mathcal{B}, \mu) = \lim_{n\to\infty} \frac{1}{n} \log \|(\mathcal{B}_x^n)^{-1}\|^{-1}$$

for μ almost every $x \in \mathcal{M}$ (see [BPe, Section 2.3], for more details).

Let $p = f^k p$ be a periodic point for f. The largest and smallest Lyapunov exponents of $\mathcal{B}$ with respect to the invariant measure μ_p on the orbit of p satisfy

$$\lambda_\pm(\mathcal{B}, \mu_p) = \frac{1}{k} \log\left(\text{the largest/smallest } |\text{eigenvalue of } \mathcal{B}_p^k|\right).$$

Since the matrices $\mathcal{A}_p^k$ and $\mathcal{B}_p^k$ are conjugate, it follows that $\lambda_\pm(\mathcal{B}, \mu_p) = \lambda_\pm(\mathcal{A}, \mu_p)$.

For the scalar cocycle ν^β, $\lambda(\nu^\beta, \mu) = \int_{\mathcal{M}} \log \nu(x)^\beta \, d\mu$ by the Birkhoff Ergodic Theorem, in particular $\lambda(\nu^\beta, \mu_p) = \frac{1}{k} \log(\nu_p^k)^\beta$.

Since the cocycle $\mathcal{A}$ is fiber bunched, there are numbers L and $\theta < 1$ such that

$$\|\mathcal{A}_x^n\| \cdot \|(\mathcal{A}_x^n)^{-1}\| \cdot (\nu_x^n)^\beta < L\, \theta^n$$

for every $x \in \mathcal{M}$ and $n \in \mathbb{N}$. It follows that

$$\lambda_+(\mathcal{A}, \mu_p) - \lambda_-(\mathcal{A}, \mu_p) + \lambda(\nu^\beta, \mu_p) = \lim_{n\to\infty} \frac{1}{n} \log(\|\mathcal{A}_p^n\| \cdot \|(\mathcal{A}_p^n)^{-1}\| \cdot (\nu_p^n)^\beta) \le \log\theta < 0,$$

and hence

$$\lambda_+(\mathcal{B}, \mu_p) - \lambda_-(\mathcal{B}, \mu_p) + \lambda(\nu^\beta, \mu_p) \le \log\theta < 0.$$

We consider the cocycle $\mathcal{F} = \mathcal{B} \oplus \nu$ over f. By [K11, Theorem 1.4], the Lyapunov exponents $\lambda_1 \le \dots \le \lambda_d$ of $\mathcal{F}$ with respect to an ergodic invariant measure μ (listed with multiplicities) can be approximated by the Lyapunov exponents of $\mathcal{F}$ at periodic points. More precisely, for any $\epsilon > 0$ there exists a periodic point $p \in \mathcal{M}$ for which the Lyapunov exponents $\lambda_1^{(p)} \le \dots \le \lambda_d^{(p)}$ of $\mathcal{F}$ satisfy $|\lambda_i - \lambda_i^{(p)}| < \epsilon$ for $i = 1, \dots, d$.

Thus for the sequence of functions $a_n(x) = \log\left(\|\mathcal{B}_x^n\| \cdot \|(\mathcal{B}_x^n)^{-1}\| \cdot (\nu_x^n)^\beta\right)$,

$$\chi(a, \mu) \overset{def}{=} \lim_{n\to\infty} \frac{a_n(x)}{n} = \lambda_+(\mathcal{B}, \mu) - \lambda_-(\mathcal{B}, \mu) + \lambda(\nu^\beta, \mu) < 0.$$

Now it follows from Lemma 4.1 that there exists N such that $a_N(x) < 0$ for all x, i.e.

$$\|\mathcal{B}_x^N\| \cdot \|(\mathcal{B}_x^N)^{-1}\| \cdot (\nu_x^N)^\beta < 1 \quad \text{for all } x \in \mathcal{M}. \tag{4.1}$$

By continuity, there exists $\tilde\theta < 1$ such that the left hand side of (4.1) is smaller than $\tilde\theta$ for all x. Writing $n \in \mathbb{N}$ as $n = mN + r$, where $m \in \mathbb{N} \cup \{0\}$ and $0 \le r < N$, we get

$$\|\mathcal{B}_x^n\| \cdot \|(\mathcal{B}_x^n)^{-1}\| \cdot (\nu_x^n)^\beta \le L\,\tilde\theta^{\,m}, \text{ where } L = \max_{x,r} \left(\|\mathcal{B}_x^r\| \cdot \|(\mathcal{B}_x^r)^{-1}\| \cdot (\nu_x^r)^\beta\right). \tag{4.2}$$

The corresponding inequality with $\hat\nu$ is obtained similarly, and we conclude that the cocycle $\mathcal{B}$ is fiber bunched. $\square$

The argument implies the following.

Corollary 4.2. *A cocycle $\mathcal{B}$ is fiber bunched if and only if there exists a number $\eta < 0$ such that for every f-periodic point $p = f^k p$,*

$$\lambda_+(\mathcal{B},\mu_p) - \lambda_-(\mathcal{B},\mu_p) + \lambda(\nu^\beta,\mu_p) = \frac{1}{k}\log\left(\frac{\text{largest } |\text{eigenvalue of } \mathcal{B}_p^k|}{\text{smallest } |\text{eigenvalue of } \mathcal{B}_p^k|}\,(\nu_p^k)^\beta\right) < \eta$$

and the corresponding enequality holds for $\hat\nu$.

4.2. **Holonomies.** An important role in our arguments is played by holonomies. We follow the notations and terminology form [V08, ASV] for linear cocycles.

Let $\mathcal{E} = \mathcal{M} \times \mathbb{R}^d$ be a trivial vector bundle over $\mathcal{M}$. We view $\mathcal{A}_x$ as a linear map from $\mathcal{E}_x$, the fiber at x, to $\mathcal{E}_{fx}$, so $\mathcal{A}_x^n : \mathcal{E}_x \to \mathcal{E}_{f^n x}$ and $\mathcal{A}_x^{-n} : \mathcal{E}_x \to \mathcal{E}_{f^{-n} x}$.

Definition 4.3. *A* stable holonomy *for a linear cocycle $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ is a continuous map $H^{\mathcal{A},s} : (x,y) \mapsto H^{\mathcal{A},s}_{x,y}$, where $x \in \mathcal{M}$, $y \in W^s(x)$, such that*

(H1) $H^{\mathcal{A},s}_{x,y}$ *is a linear map from $\mathcal{E}_x$ to $\mathcal{E}_y$;*

(H2) $H^{\mathcal{A},s}_{x,x} = Id$ *and* $H^{\mathcal{A},s}_{y,z} \circ H^{\mathcal{A},s}_{x,y} = H^{\mathcal{A},s}_{x,z}$;

(H3) $H^{\mathcal{A},s}_{x,y} = (\mathcal{A}_y^n)^{-1} \circ H^{\mathcal{A},s}_{f^n x, f^n y} \circ \mathcal{A}_x^n$ *for all $n \in \mathbb{N}$.*

Condition (H2) implies that $(H^{\mathcal{A},s}_{x,y})^{-1} = H^{\mathcal{A},s}_{y,x}$.
The unstable holonomy $H^{\mathcal{A},u}$ are defined similarly for $y \in W^u(x)$ with

(H3′) $H^{\mathcal{A},u}_{x,y} = (\mathcal{A}_y^{-n})^{-1} \circ H^{\mathcal{A},u}_{f^{-n}x, f^{-n}y} \circ \mathcal{A}_x^{-n}$ for all $n \in \mathbb{N}$.

We consider holonomies which satisfy the following Hölder condition:

(H4) $\|H^{\mathcal{A},s(u)}_{x,y} - \mathrm{Id}\,\| \le c\,\mathrm{dist}(x,y)^\beta$, where c is independent of x and $y \in W^{s(u)}_{loc}(x)$.

A local stable manifold $W^s_{loc}(x)$ is a ball in $W^s(x)$ centered at x of a small radius ρ in the intrinsic metric of $W^s(x)$. We choose ρ small enough so that (2.1) ensures that $\|Df_y\| < \nu(x)$ for all $x \in \mathcal{M}$ and $y \in W^s_{loc}(x)$. Local unstable manifolds are defined similarly, and it follows that for all $n \in \mathbb{N}$,

$$\begin{aligned} &\mathrm{dist}(f^n x, f^n y) < \nu_x^n \cdot \mathrm{dist}(x,y) \quad \text{for all } x \in \mathcal{M} \text{ and } y \in W^s_{loc}(x), \\ &\mathrm{dist}(f^{-n} x, f^{-n} y) < \hat\nu_x^{-n} \cdot \mathrm{dist}(x,y) \quad \text{for all } x \in \mathcal{M} \text{ and } y \in W^u_{loc}(x). \end{aligned} \tag{4.3}$$

Proposition 4.4. *Suppose that a cocycle $\mathcal{A}$ is fiber bunched. Then $\mathcal{A}$ has unique stable and unstable holonomies satisfying (H4). Moreover, for every $x \in \mathcal{M}$,*

$$H^{\mathcal{A},s}_{x,y} = \lim_{n\to\infty} (\mathcal{A}^n_y)^{-1} \circ \mathcal{A}^n_x, \quad y \in W^s(x), \quad \textit{and}$$
$$H^{\mathcal{A},u}_{x,y} = \lim_{n\to\infty} \left((\mathcal{A}^{-n}_y)^{-1} \circ (\mathcal{A}^{-n}_x)\right) = \lim_{n\to\infty} \left(\mathcal{A}^n_{f^{-n}y} \circ (\mathcal{A}^n_{f^{-n}x})^{-1}\right), \quad y \in W^u(x).$$

Proof. We will give the proof for the stable holonomies. The argument for the unstable holonomies is similar. Under the fiber bunching condition "at each step",

$$\|\mathcal{B}_x\| \cdot \|\mathcal{B}^{-1}_x\| \cdot \nu(x)^\beta < 1 \quad \text{for all } x \in \mathcal{M}, \tag{4.4}$$

existence of such holonomies was proved in [V08, ASV] and uniqueness in [KS13]. We indicate how to extend these results to our setting.

Since the cocycle $\mathcal{A}$ is fiber bunched (in the sense of Definition 2.2) and $\nu < 1$, there exist $N \in \mathbb{N}$ such that for all $n \geq N$ and $x \in \mathcal{M}$, $\|\mathcal{A}^n_x\| \cdot \|(\mathcal{A}^n_x)^{-1}\| \cdot (\nu^n_x)^\beta < 1$. Thus the cocycles $\mathcal{A}^n$ satisfy (4.4) and hence have unique stable holonomies.

The stable holonomies for $\mathcal{A}^N$ and $\mathcal{A}^{N+1}$ are also the stable holonomies for $\mathcal{A}^{N(N+1)}$, and hence they coincide by uniqueness. Let $H = H^{\mathcal{A}^{N+1},s} = H^{\mathcal{A}^N,s}$. Clearly, H satisfies the properties (H 1,2,4). Also, since $H^{\mathcal{A}^{N+1},s}$ and $H^{\mathcal{A}^N,s}$ satisfy (H3),

$$H_{x,y} = (\mathcal{A}^N_y)^{-1} \circ H_{f^N x,\, f^N y} \circ \mathcal{A}^N_x = (\mathcal{A}^{N+1}_y)^{-1} \circ H_{f^{N+1}x,\, f^{N+1}y} \circ \mathcal{A}^{N+1}_x.$$

Hence

$$H_{f^N x,\, f^N y} = (\mathcal{A}_{f^N y})^{-1} \circ H_{f^{N+1}x,\, f^{N+1}y} \circ \mathcal{A}_{f^N x},$$

and it follows that H satisfies (H3). The stable holonomy for $\mathcal{A}$ satisfying (H4) is unique since it is also a holonomy for $\mathcal{A}^N$. Thus $H = H^{\mathcal{A},s}$, and it remains to show that it equals the limit.

By (H3), $\mathcal{A}^n_x = (H_{f^n x, f^n y})^{-1} \circ \mathcal{A}^n_y \circ H^{\mathcal{A},s}_{x,y}$, and hence by (H4) there is a constant c_1 such that

$$\|\mathcal{A}^n_x\| = c_1 \|\mathcal{A}^n_y\| \quad \text{for all } x \in \mathcal{M}, \;\; y \in W^s_{loc}(x), \text{ and } n \in \mathbb{N}. \tag{4.5}$$

Hence

$$\|H^{\mathcal{A},s}_{x,y} - (\mathcal{A}^n_y)^{-1} \circ \mathcal{A}^n_x\| = \|(\mathcal{A}^n_y)^{-1} \circ (H^{\mathcal{A},s}_{f^n x,\, f^n y} - \mathrm{Id}) \circ \mathcal{A}^n_x\| \leq$$
$$\leq \|(\mathcal{A}^n_y)^{-1}\| \cdot \|\mathcal{A}^n_y\| \cdot c\,\mathrm{dist}(f^n x,\, f^n y)^\beta \leq c_2 \|(\mathcal{A}^n_y)^{-1}\| \cdot \|\mathcal{A}^n_y\| \cdot (\nu^n_y)^\beta \to 0$$

as $n \to \infty$ by (4.3) and fiber bunching. □

4.3. Relations between Hölder conjugacies and holonomies.

Proposition 4.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be two fiber bunched cocycles and let C be a β-Hölder continuous conjugacy between $\mathcal{A}$ and $\mathcal{B}$. Then*

(a) *C intertwines the holonomies for $\mathcal{A}$ and $\mathcal{B}$, i.e.*

$$H^{\mathcal{A},\, s(u)}_{x,y} = C(y) \circ H^{\mathcal{B},\, s(u)}_{x,y} \circ C(x)^{-1} \quad \textit{for every } x \in \mathcal{M} \textit{ and } y \in W^{s(u)}(x).$$

(b) *C conjugates the* periodic cycle functionals *of $\mathcal{A}$ and $\mathcal{B}$, i.e.*

$$H^{\mathcal{A},s}_{y,x} \circ H^{\mathcal{A},u}_{x,y} = C(x) \circ H^{\mathcal{B},s}_{y,x} \circ H^{\mathcal{B},u}_{x,y} \circ C(x)^{-1}$$

for every $x \in \mathcal{M}$ and $y \in W^s(x) \cap W^u(x)$.

(c) *C is uniquely determined by its value at one point.*

Proof. (a) Let $x \in \mathcal{M}$ and $y \in W^s(x)$. By iterating x and y forward the problem reduces to the case of $y \in W^s_{loc}(x)$. Since $\mathcal{A}(x) = C(fx) \circ \mathcal{B}_x \circ C(x)^{-1}$, we have

$$\begin{aligned}(\mathcal{A}^n_y)^{-1} \circ \mathcal{A}^n_x \;&=\; C(y) \circ (\mathcal{B}^n_y)^{-1} \circ C(f^n y)^{-1} \circ C(f^n x) \circ \mathcal{B}^n_x \circ C(x)^{-1} = \\ &= C(y) \circ (\mathcal{B}^n_y)^{-1} \circ (\mathrm{Id} + r_n) \circ \mathcal{B}^n_x \circ C(x)^{-1} = \\ &= C(y) \circ (\mathcal{B}^n_y)^{-1} \circ \mathcal{B}^n_x \circ C(x)^{-1} + C(y) \circ (\mathcal{B}^n_y)^{-1} \circ r_n \circ \mathcal{B}^n_x \circ C(x)^{-1}.\end{aligned} \tag{4.6}$$

Hölder continuity of C and (4.3) imply that

$$\begin{aligned}\|r_n\| = \|C(f^n y)^{-1} \circ C(f^n x) - \mathrm{Id}\| &\le \|C(f^n y)^{-1}\| \cdot \|C(f^n x) - C(f^n y)\| \le \\ &\le c_2 \, \mathrm{dist}(f^n x,\, f^n y)^\beta \le c_2\, (\nu^n_y)^\beta.\end{aligned}$$

Using (4.5), the above estimate, and fiber bunching of the cocycle $\mathcal{B}$, we obtain

$$\begin{aligned}\|(\mathcal{B}^n_y)^{-1} \circ r_n \circ \mathcal{B}^n_x\| &\le \|(\mathcal{B}^n_y)^{-1}\| \cdot \|r_n\| \cdot c_3\, \|\mathcal{B}^n_y\| \le \\ &\le c_4\, \|(\mathcal{B}^n_y)^{-1}\| \cdot \|\mathcal{B}^n_y\| \cdot (\nu^n_y)^\beta \le c_5 \theta^n \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Hence the second term in the last line of (4.6) tends to 0. Since $\lim_{n\to\infty} (\mathcal{A}^n_y)^{-1} \circ \mathcal{A}^n_x = H^{\mathcal{A},s}_{x,y}$ and $\lim_{n\to\infty} (\mathcal{B}^n_y)^{-1} \circ \mathcal{B}^n_x = H^{\mathcal{B},s}_{x,y}$, passing to the limit in (4.6) we obtain (a).

The statement for the unstable holonomies is proven similarly and (b) follows immediately from (a).

(c) Let $C(x_0)$ be given. By (a) for every $y \in W^s(x_0)$, the conjugacy at y is given by

$$C(y) = H^{\mathcal{B},s}_{y,x} \circ C(x_0) \circ H^{\mathcal{A},s}_{x,y}.$$

Since the stable manifold $W^s(x_0)$ is dense in $\mathcal{M}$ and C is Hölder continuous, C is uniquely determined on $\mathcal{M}$. □

4.4. **Cocycles over a diffeomorphism with a fixed point.**

Outline of the proof of Theorem 2.4. Since the cocycle $\mathcal{A}$ is fiber bunched and $\mathcal{B}$ has conjugate periodic data, $\mathcal{B}$ is also fiber bunched by Proposition 2.3. The theorem then follows from Propositions 4.6 and 4.7 below. Somewhat more directly, the argument can be outlined as follows. We consider the cocycle $\tilde{\mathcal{B}} = C(p) \circ \mathcal{B} \circ C(p)^{-1}$, so that $\tilde{\mathcal{B}}_p = \mathcal{A}_p$, and the function $\tilde{C}(q) = C(q)C(p)^{-1}$, so that $\tilde{C}(p) = \mathrm{Id}$. We construct conjugacies between $\mathcal{A}$ and $\tilde{\mathcal{B}}$ along the stable and unstable manifolds of p

$$\begin{aligned}\tilde{C}^s(x) &= H^{\mathcal{A},s}_{p,\,x} \circ H^{\tilde{\mathcal{B}},s}_{x,\,p} \quad \text{for } x \in W^s(p), \\ \tilde{C}^u(x) &= H^{\mathcal{A},u}_{p,\,x} \circ H^{\tilde{\mathcal{B}},u}_{x,\,p} \quad \text{for } x \in W^u(p).\end{aligned}$$

The proof of Proposition 4.6 shows that if x is a homoclinic point for p, i.e. $x \in W^s(p) \cap W^u(p)$, then

$$H^{\mathcal{A},s}_{x,p} \circ H^{\mathcal{A},u}_{p,x} = H^{\tilde{\mathcal{B}},s}_{x,p} \circ H^{\tilde{\mathcal{B}},u}_{p,x}, \quad \text{i.e.} \quad \tilde{C}^s(x) = \tilde{C}^u(x) \stackrel{def}{=} \tilde{C}(x).$$

The proof of Proposition 4.7 shows that $\tilde{C}$ is β-Hölder continuous on the set of homoclinic points, and hence it can be extended to $\mathcal{M}$. $C(x) = \tilde{C}(x)C(p)$ is a conjugacy between $\mathcal{A}$ and $\mathcal{B}$, and it is clear from the construction that it takes values in the closed subgroup G_0. Uniqueness follows from Proposition 4.5(c). □

Assumptions. In Propositions 4.6 and 4.7, the diffeomorphism f has a fixed point p and the cocycles $\mathcal{A}$ and $\mathcal{B}$ are fiber bunched.

Proposition 4.6. *Suppose that for each periodic point $q = f^k q$ in a neighborhood U of p there is $C(q) \in GL(d, \mathbb{R})$ such that*

$$\mathcal{A}^k_q = C(q) \circ \mathcal{B}^k_q \circ C(q)^{-1} \quad \textit{and} \quad d(C(p), C(q)) \leq c\, dist(p,q)^\beta.$$

Then $C(p)$ conjugates the periodic cycle functionals of $\mathcal{A}$ and $\mathcal{B}$ at p, i.e.

$$H^{\mathcal{A},s}_{x,p} \circ H^{\mathcal{A},u}_{p,x} = C(p) \circ H^{\mathcal{B},s}_{x,p} \circ H^{\mathcal{B},u}_{p,x} \circ C(p)^{-1} \quad \textit{for every } x \in W^s(p) \cap W^u(p).$$

The next proposition describes a sufficient condition for a conjugacy at a fixed point to extend to a conjugacy between cocycles.

Proposition 4.7. *Let $C_p \in GL(d, \mathbb{R})$ be such that*

(a) $\mathcal{A}_p = C_p \circ \mathcal{B}_p \circ C_p^{-1}$ *and*

(b) $H^{\mathcal{A},s}_{x,p} \circ H^{\mathcal{A},u}_{p,x} = C_p \circ H^{\mathcal{B},s}_{x,p} \circ H^{\mathcal{B},u}_{p,x} \circ C_p^{-1}$ *for every $x \in W^s(p) \cap W^u(p)$.*

Then there exists a unique β-Hölder continuous conjugacy $C(x)$ between $\mathcal{A}$ and $\mathcal{B}$ such that $C(p) = C_p$. Moreover, if $\mathcal{A}$ and $\mathcal{B}$ take values in a closed subgroup G_0 of $GL(d, \mathbb{R})$ and $C_p \in G_0$, then $C(x) \in G_0$ for all x.

We note that the first assumption on C_p is obviously necessary, and so is the second one by Proposition 4.5 (b). Thus a conjugacy C_p between the matrices $\mathcal{A}_p$ and $\mathcal{B}_p$ extends to a conjugacy between cocycles if and only if (b) is satisfied.

Proof of Proposition 4.6. First we modify the cocycle $\mathcal{B}$ so that the two cocycles coincide at the fixed point p. We define the cocycle $\tilde{\mathcal{B}}$ and the function $\tilde{C}(q)$ by

$$\tilde{\mathcal{B}}_x = C(p) \circ \mathcal{B}_x \circ C(p)^{-1} \quad \text{and} \quad \tilde{C}(q) = C(q)C(p)^{-1}, \quad q \in U.$$

The cocycle $\tilde{\mathcal{B}}$ is fiber bunched and $\tilde{\mathcal{B}}_p = \mathcal{A}_p$. Also, $\mathcal{A}^k_q = \tilde{C}(q) \circ \tilde{\mathcal{B}}^k_q \circ \tilde{C}(q)^{-1}$ and

$$d(\tilde{C}(q), \mathrm{Id}) \leq \tilde{c}\, \mathrm{dist}(p,q)^\beta \;\text{ for all } q \in U.$$

We prove that for every $x \in W^s(p) \cap W^u(p)$,

$$H^{\tilde{\mathcal{B}},u}_{p,\,x} \circ H^{\tilde{A},u}_{x,\,p} \circ H^{\tilde{A},s}_{p,\,x} \circ H^{\tilde{B},s}_{x,\,p} = \mathrm{Id}.$$

By Proposition 4.4, $H^{\tilde{\mathcal{B}}} = C(p) \circ H^{\mathcal{B}} \circ C(p)^{-1}$, and Proposition 4.6 follows.

In the rest of the proof, we write $\mathcal{B}$ for $\tilde{\mathcal{B}}$ and C for $\tilde{C}$ to simplify the notations, and we fix $x \in W^s(p) \cap W^u(p)$. By Proposition 4.4,

$$H^{\mathcal{A},s}_{p,x} \circ H^{\mathcal{B},s}_{x,p} = \lim_{n\to\infty} \left((\mathcal{A}^n_x)^{-1} \circ \mathcal{A}^n_p \circ (\mathcal{B}^n_p)^{-1} \circ \mathcal{B}^n_x \right) = \lim_{n\to\infty} \left((\mathcal{A}^n_x)^{-1} \circ \mathcal{B}^n_x \right)$$

since $\mathcal{B}^n_p = \mathcal{A}^n_p$. Similarly,

$$H^{\mathcal{B},u}_{p,x} \circ H^{\mathcal{A},u}_{x,p} = \lim_{n\to\infty} \left(\mathcal{B}^n_{f^{-n}x} \circ (\mathcal{A}^n_{f^{-n}x})^{-1} \right).$$

Thus,

$$H^{\mathcal{B},u}_{p,x} \circ H^{\mathcal{A},u}_{x,p} \circ H^{\mathcal{A},s}_{p,x} \circ H^{\mathcal{B},s}_{x,p} = \lim_{n\to\infty} \left(\mathcal{B}^n_{f^{-n}x} \circ (\mathcal{A}^n_{f^{-n}x})^{-1} \circ (\mathcal{A}^n_x)^{-1} \circ \mathcal{B}^n_x \right),$$

and we will show that the limit on the right hand side equals the identity.

Since $x \in W^s(p) \cap W^u(p)$, by (2.1) there is a constant $c_1 = c_1(x)$ such that

$$\text{dist}(f^n x,\, p) < \nu^n_x \cdot c_1 \text{dist}_{W^s(p)}(x,p) \;\text{ and }\; \text{dist}(f^{-n}x,\, p) < \hat{\nu}^{-n}_x \cdot c_1 \text{dist}_{W^u(p)}(x,p),$$

and hence

$$\text{dist}(f^n x,\; f^{-n}x) < c_2 \max\{\nu^n_x, \hat{\nu}^{-n}_x\}.$$

Therefore, for all sufficiently large n we can apply Anosov Closing Lemma to the orbit segment $\{f^i(x),\; i = -n, \dots, n\}$ [KtH, Theorem 6.4.15]. Thus there exists a periodic point $q = f^{2n}q$ such that

$$\text{dist}(f^i x,\; f^i q) \le c_3 \max\{\nu^n_x, \hat{\nu}^{-n}_x\} \quad \text{for } i = -n, \dots, n.$$

Additionally, we assume that n is large enough so that $f^{-n}q \in U$.

Now we express $\mathcal{B}^n_{f^{-n}x}$, $(\mathcal{A}^n_{f^{-n}x})^{-1} \circ (\mathcal{A}^n_x)^{-1}$, and $\mathcal{B}^n_x$ in terms of the values of the cocycles at the corresponding iterates of q. To use the holonomies, we consider the point

$$z = W^s_{loc}(q) \cap W^u_{loc}(x).$$

It is easy to see that for $i = -n, \dots, n$,

$$\text{dist}(f^i z,\; f^i x) \le c_4 \max\{\nu^n_x,\; \hat{\nu}^{-n}_x\} \;\text{ and }\; \text{dist}(f^i z,\; f^i q) \le c_4 \max\{\nu^n_x,\; \hat{\nu}^{-n}_x\}. \tag{4.7}$$

Since $f^i z \in W^u_{loc}(f^i x)$ and $f^i z \in W^s_{loc}(f^i q)$, by the properties (H3) and (H3′) we have

$$\mathcal{B}^n_x = H^{\mathcal{B},u}_{f^n z,\, f^n x} \circ \mathcal{B}^n_z \circ H^{\mathcal{B},u}_{x,\, z} = H^{\mathcal{B},u}_{f^n z,\, f^n x} \circ H^{\mathcal{B},s}_{f^n q,\, f^n z} \circ \mathcal{B}^n_q \circ H^{\mathcal{B},s}_{z,\, q} \circ H^{\mathcal{B},u}_{x,\, z}.$$

It follows from (H4) that

$$H^{s,\mathcal{B}}_{z,q} = \text{Id} + R^{s,\mathcal{B}}_{z,q}, \quad \text{where } \; \|R^{s,\mathcal{B}}_{z,q}\| \le c\, \text{dist}(z,q)^\beta \le c_5 (\max\{\nu^n_x,\; \hat{\nu}^{-n}_x\})^\beta,$$

and similar estimates hold for the other holonomies due to (4.7). Thus we obtain

$$\mathcal{B}^n_x = (\text{Id} + R^n_1) \circ \mathcal{B}^n_q \circ (\text{Id} + R^n_2), \quad \text{where } \|R^n_1\|, \|R^n_2\| \le c_6 (\max\{\nu^n_x, \hat{\nu}^{-n}_x\})^\beta. \tag{4.8}$$

Similarly,

$$\mathcal{B}^n_{f^{-n}x} = (\text{Id} + R^n_3) \circ \mathcal{B}^n_{f^{-n}q} \circ (\text{Id} + R^n_4) \tag{4.9}$$

$$(\mathcal{A}^n_{f^{-n}x})^{-1} \circ (\mathcal{A}^n_x)^{-1} = (\mathcal{A}^{2n}_{f^{-n}x})^{-1} = (\text{Id} + R^n_5) \circ (\mathcal{A}^{2n}_{f^{-n}q})^{-1} \circ (\text{Id} + R^n_6). \tag{4.10}$$

Since $f^{-n}q$ is a point of period $2n$ in the neighborhood U of p, by the assumption there exists $C(f^{-n}q)$ such that

$$\begin{aligned} &\mathcal{A}^{2n}_{f^{-n}q} = C(f^{-n}q) \circ \mathcal{B}^{2n}_{f^{-n}q} \circ C(f^{-n}q)^{-1}, \quad \text{where} \\ &C(f^{-n}q) = \mathrm{Id} + R^n_7 \quad \text{ and } C(f^{-n}q)^{-1} = \mathrm{Id} + R^n_8 \quad \text{ with} \\ &\|R^n_7\|,\ \|R^n_8\|, \le c_7 \operatorname{dist}(p, f^{-n}q)^\beta \le c_8(\max\{\nu^n_x, \hat\nu^{-n}_x\})^\beta. \end{aligned} \tag{4.11}$$

Using (4.10) and (4.11) and combining terms of type $\mathrm{Id} + R^n_i$, we obtain

$$(\mathcal{A}^n_{f^{-n}x})^{-1} \circ (\mathcal{A}^n_x)^{-1} = (\mathrm{Id} + R^n_9) \circ (\mathcal{B}^{2n}_{f^{-n}q})^{-1} \circ (\mathrm{Id} + R^n_{10}) \tag{4.12}$$

Finally (4.8), (4.9), and (4.12) yield

$$\mathcal{B}^n_{f^{-n}x} \circ (\mathcal{A}^n_{f^{-n}x})^{-1} \circ (\mathcal{A}^n_x)^{-1} \circ \mathcal{B}^n_x = \tag{4.13}$$

$$\begin{aligned} &= (\mathrm{Id} + R^n_3) \circ \mathcal{B}^n_{f^{-n}q} \circ (\mathrm{Id} + R^n_{11}) \circ (\mathcal{B}^n_{f^{-n}q})^{-1} \circ (\mathcal{B}^n_q)^{-1} \circ (\mathrm{Id} + R^n_{12}) \circ \mathcal{B}^n_q \circ (\mathrm{Id} + R^n_2) \\ &= \mathrm{Id} + \mathcal{B}^n_{f^{-n}q} \circ R^n_{11} \circ (\mathcal{B}^n_{f^{-n}q})^{-1} + (\mathcal{B}^n_q)^{-1} \circ R^n_{12} \circ \mathcal{B}^n_q + \\ &+ \mathcal{B}^n_{f^{-n}q} \circ R^n_{11} \circ (\mathcal{B}^n_{f^{-n}q})^{-1} \circ (\mathcal{B}^n_q)^{-1} \circ R^n_{12} \circ \mathcal{B}^n_q \ + \ \text{smaller terms,} \end{aligned}$$

where $\|R^n_i\| \le c_9(\max\{\nu^n_x, \hat\nu^{-n}_x\})^\beta$. Since the cocycle $\mathcal{B}$ is fiber bunched,

$$\|(\mathcal{B}^n_q)^{-1}\| \cdot \|\mathcal{B}^n_q\| \cdot \|R^n_i\| \le c_{10}\, \theta^n \quad \text{and} \quad \|\mathcal{B}^n_{f^{-n}q}\| \cdot \|(\mathcal{B}^n_{f^{-n}q})^{-1}\| \cdot \|R^n_i\| \le c_{11}\, \theta^n.$$

Thus we conclude that

$$\mathcal{B}^n_{f^{-n}x} \circ (\mathcal{A}^n_{f^{-n}x})^{-1} \circ (\mathcal{A}^n_x)^{-1} \circ \mathcal{B}^n_x = \mathrm{Id} + R^n, \quad \text{where } \|R^n\| \le c_{12}\, \theta^n \to 0 \text{ as } n \to \infty,$$

and hence

$$H^{\mathcal{B},u}_{p,x} \circ H^{\mathcal{A},u}_{x,p} \circ H^{\mathcal{A},s}_{p,x} \circ H^{\mathcal{B},s}_{x,p} = \mathrm{Id}.$$

This completes the proof of Proposition 4.6. □

Proof of Proposition 4.7. We define a conjugacy C^s on the stable manifold of p,

$$C^s(x) = H^{\mathcal{A},s}_{p,x} \circ C_p \circ H^{\mathcal{B},s}_{x,p} \quad \text{for } x \in W^s(p). \tag{4.14}$$

Clearly, $C(p) = C_p$. Also,

$$\begin{aligned} \mathcal{A}^n_x &= H^{\mathcal{A},s}_{p,f^nx} \circ \mathcal{A}^n_p \circ H^{\mathcal{A},s}_{x,p} = H^{\mathcal{A},s}_{p,f^nx} \circ C_p \circ \mathcal{B}^n_p \circ C_p^{-1} \circ H^{\mathcal{A},s}_{x,p} = \\ &= H^{\mathcal{A},s}_{p,f^nx} \circ C_p \circ H^{\mathcal{B},s}_{f^nx,p} \circ \mathcal{B}^n_x \circ H^{\mathcal{B},s}_{p,x} \circ C_p^{-1} \circ H^{\mathcal{A},s}_{x,p} \ = \ C^s(f^nx) \circ \mathcal{B}^n_x \circ C^s(x)^{-1}. \end{aligned}$$

Similarly, we define a conjugacy C^u along the unstable manifold of p,

$$C^u(x) = H^{\mathcal{A},u}_{p,x} \circ C_p \circ H^{\mathcal{B},u}_{x,p} \quad \text{for } x \in W^u(p).$$

Let $X = W^u(p) \cap W^s(p)$ be the set of homoclinic points of p. By the assumption (b),

$$C^s(x) = C^u(x) \overset{def}{=} C(x) \quad \text{for every}\, x \in X. \tag{4.15}$$

The set of homoclinic points of p is known to be dense in $\mathcal{M}$ [Bo]. To extend the function C from X to $\mathcal{M}$, we show that C is Hölder continuous on X. Let x and y be two sufficiently close points in X. We note that the distances between x and y along $W^s(p)$ and $W^u(p)$ can be large. To make an estimate we consider the point

$$z = W^u_{loc}(x) \cap W^s_{loc}(y),$$

which is also in X. By the definition of $C = C^s$ and properties of holonomies,

$$C(z) = H^{\mathcal{A},s}_{y,z} \circ C(y) \circ H^{\mathcal{B},s}_{z,y} = (\mathrm{Id} + R^{\mathcal{A}}_{y,z}) \circ C(y) \circ (\mathrm{Id} + R^{\mathcal{B}}_{y,z}),$$

where $\|R^{\mathcal{A}}_{y,z}\|,\ \|R^{\mathcal{B}}_{y,z}\| \le c\,\mathrm{dist}(y,z)^\beta$. Hence

$$\begin{aligned} C(z) \circ C(y)^{-1} &= (\mathrm{Id} + R^{\mathcal{A},s}_{y,z}) \circ C(y) \circ (\mathrm{Id} + R^{\mathcal{B},s}_{y,z}) \circ C(y)^{-1} = \\ &= \mathrm{Id} + R^{\mathcal{A},s}_{y,z} + C(y) \circ R^{\mathcal{B},s}_{y,z} \circ C(y)^{-1} + R^{\mathcal{A},s}_{y,z} \circ C(y) \circ R^{\mathcal{B},s}_{y,z} \circ C(y)^{-1}. \end{aligned} \tag{4.16}$$

Similarly, using unstable holonomies, we obtain

$$\begin{aligned} C(x) \circ C(z)^{-1} &= \\ &= \mathrm{Id} + R^{\mathcal{A},u}_{x,z} + C(z) \circ R^{\mathcal{B},u}_{x,z} \circ C(z)^{-1} + R^{\mathcal{A},u}_{x,z} \circ C(z) \circ R^{\mathcal{B},u}_{x,z} \circ C(z)^{-1}. \end{aligned} \tag{4.17}$$

Now we show that $\|C\|$ and $\|C^{-1}\|$ are bounded on X. We fix a small number ϵ and choose a finite subset Y of X such that for each $x \in X$ there is $y \in Y$ such that $\mathrm{dist}(x,y) \le \epsilon$. Since Y is finite, there is a constant M such that

$$\|C(y)\| \le M \quad \text{and} \quad \|C(y)^{-1}\| \le M \quad \text{for all } y \in Y.$$

Let $x \in X$, let $y \in Y$ be such that $\mathrm{dist}(x,y) \le \epsilon$, and let $z = W^s_{loc}(x) \cap W^u_{loc}(y)$. Then multiplying both sides of (4.16) by $C(y)$ and estimating the norm we see that

$$\|C(z)\| \le (2 + 2M^2)M,$$

assuming that ϵ is sufficiently small so that $c\,\mathrm{dist}(x,z)^\beta < 1$. Now boundedness of $\|C(x)\|$ follows similarly from (4.17). One can obtain expressions for $C(z)^{-1} \circ C(y)$ and $C(x)^{-1} \circ C(z)$ similar to (4.16) and (4.17) and conclude that $\|C(x)^{-1}\|$ is also bounded on X.

Now it follows from (4.16) and (4.17) that for any sufficiently close x, y in X

$$C(x) \circ C(y)^{-1} = C(x) \circ C(z)^{-1} \circ C(z) \circ C(y)^{-1} = \mathrm{Id} + R_{x,y},$$
$$\text{where} \quad \|R_{x,y}\| \le c'\mathrm{dist}(x,y)^\beta,$$

and hence

$$\begin{aligned} d(C(x), C(y)) &= \|C(x) - C(y)\| + \|C(x)^{-1} - C(y)^{-1}\| \le \\ &\le \|C(x)C(y)^{-1} - \mathrm{Id}\| \cdot \|C(y)\| + \|C(x)^{-1}\| \cdot \|\mathrm{Id} - C(x)C(y)^{-1}\| \le \\ &\le 2c'M'\,\mathrm{dist}(x,y)^\beta. \end{aligned} \tag{4.18}$$

Thus we can extend the function C on X to a β-Hölder continuous function on $\mathcal{M}$, and

$$\mathcal{A}^n_x = C(f^n x) \circ \mathcal{B}^n_x \circ C(x)^{-1} \quad \text{for all } x \in \mathcal{M} \text{ and } n \in \mathbb{Z}.$$

The conjugacy C takes values in the closed subgroup G_0 by the construction: the holonomies take values in G_0 by Proposition 4.4, hence so does the restriction of C to X by (4.14) and (4.15), and thus so does C. Uniqueness of the conjugacy follows from Proposition 4.5(c). ☐

4.5. **Centralizers of cocycles and connections to conjugacies.**

The *centralizer* of a cocycle of $\mathcal{A}$ is the set

$$Z(\mathcal{A}) = \{D : \mathcal{M} \to GL(d,\mathbb{R}) \mid \mathcal{A}_x = D(fx) \circ \mathcal{A}_x \circ D(x)^{-1} \quad \text{for all } x \in \mathcal{M}\}.$$

We consider the centralizer in the β-Hölder category.

It is easy to see that $Z(\mathcal{A})$ is a group with respect to pointwise multiplication and that $Z(\mathcal{A})$ is a subgroup of $Z(\mathcal{A}^k)$ for all $k \geq 1$.

Proposition 4.8. *For any fiber bunched cocycle $\mathcal{A}$ there exists $M \geq 1$ such that*

$$Z(\mathcal{A}^{MT}) = Z(\mathcal{A}^M) \ \text{ for all } \ T \geq 1.$$

Proof. We note that for every $k \geq 1$ the cocycle $\mathcal{A}^k$ are also fiber bunched.

Let p be a periodic point of f of period N. Then it is a fixed point for f^N, and we consider the iterate $\bar{\mathcal{A}} = \mathcal{A}^N$ over f^N. An element D of the centralizer of $\bar{\mathcal{A}}^k$ is a conjugacy between $\bar{\mathcal{A}}^k$ and itself. Hence by Proposition 4.5, D is uniquely determined by its value at p. By Proposition 4.7, a matrix $D_p = D(p)$ extends to a Hölder conjugacy D on $\mathcal{M}$ if and only if

$$\bar{\mathcal{A}}_p^k = D_p \circ \bar{\mathcal{A}}_p^k \circ D_p^{-1} \quad \text{and}$$
$$H_{x,p}^{\bar{\mathcal{A}}^k,s} \circ H_{p,x}^{\bar{\mathcal{A}}^k,u} = D_p \circ H_{x,p}^{\bar{\mathcal{A}}^k,s} \circ H_{p,x}^{\bar{\mathcal{A}}^k,u} \circ D_p^{-1} \ \text{ for every } x \in W^s(p) \cap W^u(p).$$

The second condition is the same for all $k \geq 1$ since the holonomies of $\bar{\mathcal{A}}$ coincide with the holonomies of $\bar{\mathcal{A}}^k$ by the uniqueness.

The first condition is equivalent to the system of linear equations $\bar{\mathcal{A}}_p^k \circ D_p = D_p \circ \bar{\mathcal{A}}_p^k$ in d^2 variables, and hence the set of its solutions can be identified with a subspace V_k of $\mathbb{R}^{d^2}$. Intersecting this set with $GL(d,\mathbb{R})$ gives the centralizer of the matrix $\bar{\mathcal{A}}_p^k$. The dimensions of the subspaces V_k are bounded by d^2. Let $L \geq 1$ be the smallest number such that $\dim V_L = \max_k \dim V_k$. Clearly $V_L \subseteq V_{LT}$, and hence $V_L = V_{LT}$. Therefore,

$$Z(\bar{\mathcal{A}}^{LT}) = Z(\bar{\mathcal{A}}^L) \quad \text{i.e.} \quad Z(\mathcal{A}^{NL\cdot T}) = Z(\mathcal{A}^{NL}) \quad \text{for all } \ T \geq 1.$$

☐

The following proposition is easy to verify.

Proposition 4.9. *Let $\mathcal{C}(\mathcal{A},\mathcal{B})$ be the set of conjugacies between $\mathcal{A}$ and $\mathcal{B}$, and let $C_1 \in \mathcal{C}(\mathcal{A},\mathcal{B})$. Then $C_2 \in \mathcal{C}(\mathcal{A},\mathcal{B})$ if and only if $C_1C_2^{-1} \in Z(\mathcal{A})$. Thus the conjugacy between $\mathcal{A}$ and $\mathcal{B}$ is unique up to an element of the centralizer and $\mathcal{C}(\mathcal{A},\mathcal{B}) = Z(\mathcal{A})C_1$.*

4.6. **Proof of Theorem 2.2.** To obtain a fixed point, we pass to an iterate of f. Let p_1 be a periodic point of f of period N. We consider the diffeomorphism f^N and the cocycles $\mathcal{A}^N$ and $\mathcal{B}^N$ over f^N. Clearly these cocycles are β-Hölder continuous and fiber bunched. Thus we apply Theorem 2.4 with $C(p) = \text{Id}$ and conclude that there exists a β-Hölder continuos conjugacy C_1 between $\mathcal{A}^N$ and $\mathcal{B}^N$. It remains show that there exists a conjugacy beween the original cocycles $\mathcal{A}$ and $\mathcal{B}$ over f.

By Proposition 4.8 there exists M such that $Z(\mathcal{A}^{NM\cdot T}) = Z(\mathcal{A}^{NM})$ for every $T \geq 1$. We note that C_1 is also a conjugacy for $\mathcal{A}^{NM}$ and $\mathcal{B}^{NM}$.

It is known that any transitive Anosov diffeomorphism has periodic points of all sufficiently large periods. We pick a periodic point p_2 of a period $K > 1$ relatively prime with MN. As above, we obtain a conjugacy C_2 for the cocycles $\mathcal{A}^K$ and $\mathcal{B}^K$ over f^K. Thus both C_1 and C_2 are Hölder conjugacies for the cocycles $\mathcal{A}^{NMK}$ and $\mathcal{B}^{NMK}$ over f^{NMK}, and hence by Proposition 4.9, $C_1 C_2^{-1} \in Z(\mathcal{A}^{NMK}) = Z(\mathcal{A}^{NM})$. Since C_1 is a conjugacy for $\mathcal{A}^{NM}$ and $\mathcal{B}^{NM}$, C_2 is also a conjugacy for these cocycles.

Thus C_2 is a conjugacy for the cocycles over f^{NM} and f^K, where MN and K are relatively prime. Hence there exist integers r and s such that $NMr + Ks = 1$, and it is easy to see that C_2 is also a conjugacy for the cocycles $\mathcal{A}$ and $\mathcal{B}$ over f.

This completes the proof of the theorem. □

4.7. **Proof of Theorem 2.7.** Since the cocycle $\mathcal{B}$ is uniformly quasiconformal (see Definition 2.6), it satisfies the fiber bunching condition (2.2) with

$$L = \sup_{x,n} \left(\|\mathcal{B}_x^n\| \cdot \|(\mathcal{B}_x^n)^{-1}\| \right) \quad \text{and} \quad \theta = \max_x \nu(x).$$

Let C be a μ-measurable conjugacy between $\mathcal{A}$ and $\mathcal{B}$. First we show that C intertwines holonomies of $\mathcal{A}$ and $\mathcal{B}$ on a set of full measure, i.e. there exists a set $Y \subset \mathcal{M}$, $\mu(Y) = 1$, such that

$$H_{x,y}^{\mathcal{A},s} = C(y) \circ H_{x,y}^{\mathcal{B},s} \circ C(x)^{-1} \quad \text{for all } x, y \in Y \text{ such that } y \in W^s(x), \tag{4.19}$$

and a similar statement holds for the unstable holonomies.

Let $x \in \mathcal{M}$ and $y \in W^s(x)$. As in the proof of Proposition 4.5(a), we obtain that

$$(\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n = C(y) \circ (\mathcal{B}_y^n)^{-1} \circ \mathcal{B}_x^n \circ C(x)^{-1} + C(y) \circ (\mathcal{B}_y^n)^{-1} \circ r_n \circ \mathcal{B}_x^n \circ C(x)^{-1}, \tag{4.20}$$

where

$$\|r_n\| \leq \|C(f^n y)^{-1}\| \cdot \|C(f^n x) - C(f^n y)\|.$$

Since C is μ-measurable, by Lusin's theorem there exists a compact set $S \subset \mathcal{M}$ with $\mu(S) > 1/2$ such that C is uniformly continuous on S and hence $\|C\|$ and $\|C^{-1}\|$ are bounded on S. Let Y be the set of points in $\mathcal{M}$ for which the frequency of visiting S equals $\mu(S) > 1/2$. By Birkhoff Ergodic Theorem $\mu(Y) = 1$. If x and y are in Y, there exists a sequence $\{n_i\}$ such that $f^{n_i}x$ and $f^{n_i}y$ are in Y for all i. It follows that

$$\|r_{n_i}\| \to 0 \quad \text{as } i \to \infty$$

and $\|C\|$, $\|C^{-1}\|$ are uniformly bounded on $\{x_{n_i},\, y_{n_i}\}$. The product

$$\|(\mathcal{B}_y^n)^{-1}\| \cdot \|\mathcal{B}_x^n\| \le \|H^{\mathcal{B},s}_{f^n x,\, f^n y}\| \cdot \|(\mathcal{B}_x^n)^{-1}\| \cdot \|H^{\mathcal{B},s}_{x,y}\| \cdot \|\mathcal{B}_x^n\|$$

is uniformly bounded since the cocycle $\mathcal{B}$ is uniformly quasiconformal.

Thus for every x and y in Y such that $y \in W^s(x)$, the second term in (4.20) tends to 0 along a subsequence, and (4.19) follows. The statement for the unstable holonomies is proven similarly.

Let $x, y \in Y$ and $y \in W^s_{loc}(x)$. Then by (4.19)

$$C(y) = H^{\mathcal{A},s}_{x,y} \circ C(x) \circ H^{\mathcal{B},s}_{x,y}.$$

It follows as in the proof of Proposition 4.7, (4.16), that

$$C(y) \circ C(x)^{-1} = \; \mathrm{Id} + R^{\mathcal{A},s}_{x,y} + C(x) \circ R^{\mathcal{B},s}_{y,z} \circ C(x)^{-1} + R^{\mathcal{A},s}_{y,z} \circ C(x) \circ R^{\mathcal{B},s}_{x,y} \circ C(x)^{-1},$$

where

$$\|R^{\mathcal{A}}_{x,y}\|,\; \|R^{\mathcal{B}}_{x,y}\| \le c\, \mathrm{dist}(x,z)^\beta.$$

Since C is bounded on Y, this implies that

$$\|C(y) \circ C(x)^{-1} - \mathrm{Id}\| \le c_1\, \mathrm{dist}(x,y)^\beta,$$

and it follows as in (4.18) that

$$d(C(x), C(y)) \le c_2\, \mathrm{dist}(x,y)^\beta, \tag{4.21}$$

where c_2 does not depend on x and y. The same holds for any $x, y \in Y$ such that $y \in W^u_{loc}(x)$.

We consider a small open set U in $\mathcal{M}$ with a product structure, i.e.

$$U = W^s_{loc}(x_0) \times W^u_{loc}(x_0) \overset{def}{=} \{W^s_{loc}(x) \cap W^u_{loc}(y) \mid x \in W^s_{loc}(x_0),\; y \in W^u_{loc}(x_0)\}.$$

Since the measure μ has local product structure, μ is equivalent to the product of conditional measures on $W^s_{loc}(x_0)$ and $W^u_{loc}(x_0)$, and hence for μ almost all local stable leaves in U, the set of points of Y on the leaf has full conditional measure. Since μ has full support, the conditional measures on almost all leaves have full support.

Hence for any two points x and z in $Y \cap U$ that lie on two such stable leaves, there exists a point $y \in W^s_{loc}(x) \cap Y$ such that $W^u_{loc}(y) \cap W^s_{loc}(z)$ is also in $Y \cap U$. It follows from (4.21) and the local product structure of the stable and unstable manifolds that

$$d(C(x), C(z)) \le c_3\, \mathrm{dist}(x,z)^\beta.$$

This estimate holds for all x, z in a set of full measure $\tilde{Y} \subset Y$.

Let $\bar{Y} = \bigcap_{n=-\infty}^{\infty} f^n(\tilde{Y})$. Then $\bar{Y}$ is f-invariant and $A(x) = C(fx) \circ \mathcal{B}(x) \circ C(x)^{-1}$ for all $x \in \bar{Y}$. Since μ has full support and $\mu(\bar{Y}) = 1$, the set $\tilde{Y}$ is dense in $\mathcal{M}$. Hence we can extend C from $\bar{Y}$ and obtain a Hölder continuous conjugacy $\tilde{C}$ on $\mathcal{M}$ that coincides with C on a set of full measure.

5. An application: smooth conjugacy to a small perturbation for Anosov automorphisms

Let g be an Anosov diffeomorphism of $\mathcal{M}$. If f is a diffeomorphism of $\mathcal{M}$ sufficiently C^1 close to g, then f is also Anosov and it is topologically conjugate to g, i.e. there exists a homeomorphism h of $\mathcal{M}$ such that

$$g = h^{-1} \circ f \circ h.$$

Moreover, the conjugacy is unique when chosen near identity. (See, e.g. [KtH, Corollary 18.2.2]). The conjugacy h is only Hölder continuous in general, and it is important to find out when the diffeomorphisms f and g are *smoothly* conjugate. If h is a C^1 diffeomorphism, then the derivatives of the return maps of f and g at the corresponding periodic points are conjugate. Indeed, differentiating $g^n = h^{-1} \circ f^n \circ h$ at periodic points $p = f^n(p)$ yields

$$D_p g^n = (D_p h)^{-1} \circ D_{h(p)} f^n \circ D_p h \quad \text{whenever} \;\; p = f^n(p).$$

A diffeomorphism g is said to be *locally rigid* if for any C^1-small perturbation f the conjugacy of the derivatives at the periodic points is sufficient for h to be C^1. The problem of local rigidity has been extensively studied and Anosov diffeomorphisms with one-dimensional stable and unstable distributions were shown to be locally rigid [dlL87, dlLM88, dlL92]. In general, this is not the case for systems with higher-dimensional distributions [dlL92, dlL02]. Positive results were established for certain classes of diffeomorphisms that are conformal on the full stable and unstable distributions, [dlL02, KS03, dlL04, KS09]. In a different direction, local rigidity was proved in [G08] for an irreducible Anosov toral automorphism $L : \mathbb{T}^d \to \mathbb{T}^d$ with real eigenvalues of distinct moduli, as well as for some nonlinear systems with similar structure. Recently, this result was extended to a broad class of Anosov automorphisms.

Theorem 5.1. [GKS11] *Let $L : \mathbb{T}^d \to \mathbb{T}^d$ be an irreducible Anosov automorphism such that no three of its eigenvalues have the same modulus. Let f be a C^1-small perturbation of L such that the derivative $D_p f^n$ is conjugate to L^n whenever $f^n(p) = p$. Then f is $C^{1+Hölder}$ conjugate to L.*

We recall that an automorphism L is called to be *irreducible* if it has no rational invariant subspaces, or equivalently if its characteristic polynomial is irreducible over $\mathbb{Q}$. Examples in [G08] show that irreducibility of L is a necessary assumption for local rigidity except when L is conformal on the stable and unstable distributions.

Theorem 2.4 allows us to obtain an alternative sufficient condition for smoothness of the conjugacy to a small perturbation. Instead of the assumption on the eigenvalues of L we make an assumption that the conjugacy of the periodic data of the cocycles $L = DL$ and Df is Hölder continuous at a single periodic point.

Theorem 5.2. *Let $L : \mathbb{T}^d \to \mathbb{T}^d$ be an irreducible Anosov automorphism and let f be a C^1-small perturbation of L. Suppose that for each periodic point $p = f^n(p)$ there is $C(p)$ such that $D_p f^n = C(p) \circ L^n \circ C(p)^{-1}$ and $C(p)$ is Hölder continuous at a periodic point p_0. Then f is $C^{1+\text{Hölder}}$ conjugate to L.*

The proof of this theorem differs from the proof of Theorem 5.1 only in the way we obtain conformality of Df on certain invariant sub-bundles, as explained below.

We denote by $E^{u,L}$ the unstable distribution of L. Let $1 < \rho_1 < \rho_2 < \cdots < \rho_l$ be the distinct moduli of the unstable eigenvalues of L, and let

$$E^{u,L} = E_1^L \oplus E_2^L \oplus \cdots \oplus E_l^L$$

be the corresponding splitting of the unstable distribution. Since f is C^1 close to L, f is also Anosov, and its unstable distribution $E^{u,f}$ splits into a direct sum of l invariant Hölder continuous distributions close to the corresponding distributions for L:

$$E^{u,f} = E_1^f \oplus E_2^f \oplus \cdots \oplus E_l^f$$

(see, e.g. [Pe04, Section 3.3]).

Let $\mathcal{A} = L|E_i^L$ and $\mathcal{B} = Df|E_i^f$. Conformality of the cocycles plays an important role in establishing smoothness of the conjugacy. Since L is irreducible, all its eigenvalues are simple. Thus the restriction of L to E_i^L is diagonalizable over $\mathbb{C}$ and its eigenvalues are of the same modulus. Hence the cocycle $\mathcal{A} = L|E_i^L$ is conformal in some norm.

In [GKS11], conformality of $\mathcal{B}$ at the periodic points together with the assumption that the distributions E_i^L and E_i^f are either one- or two-dimensional allows us to conclude that, by [KS10, Theorem 1.3], the cocycle $\mathcal{B}$ is conformal. In higher dimensions, conformality at the periodic points does not imply conformality [KS10, Proposition 1.2]. In Theorem 5.2 we make no assumptions on the dimensions of E_i^L, and so we use a different approach to obtain conformality of $\mathcal{B}$.

Let $\beta > 0$ be so that $C(p)$ is β-Hölder at p_0 and all cocycles $Df|E_i^f$ are β-Hölder. Since the cocycle $\mathcal{A} = L|E_i^L$ is conformal, it is fiber bunched and it follows that the cocycle $\mathcal{B} = Df|E_i^f$ is also fiber bunched. We consider the iterates $\mathcal{A}^N$ and $\mathcal{B}^N$ over f^N, where N is the period of p_0. Theorem 2.4 implies that there exists a Hölder continuos conjugacy C between $\mathcal{A}^N$ and $\mathcal{B}^N$. Since $\mathcal{A}^N$ is conformal, this implies that $\mathcal{B}^N$ is uniformly quasiconformal, and hence so is $\mathcal{B}$. By [KS13, Corollary 3.2], $\mathcal{B}$ is conformal with respect to a continuous Riemannian metric on E_i^f.

After conformality of Df on each sub-bundle E_i^f is obtained, the proof of Theorem 5.2 proceeds exactly as the proof of Theorem 5.1. We consider the topological conjugacy h between L and f close to the identity. We use conformality to show that h is $C^{1+\text{Hölder}}$ along the leaves of the linear foliation tangent to E_i^L, and then we establish the smoothness of h on $\mathcal{M}$. □

Corollary 2.5, which we prove next, shows that the cocycles L and Df are Hölder cohomologous without irreducibility assumption on L. This is not known to imply smoothness of h. The arguments in Theorems 5.1 and 5.2 use both density of the subspaces E_i^L in $\mathbb{T}^d$ and conformality of $L|E_i^L$, which follow from irreducibility.

Proof of Corollary 2.5. The proof is closely related to the above argument.

Let $A(x) = A$ be the generator of $\mathcal{A}$. Let $\rho_1 < \cdots < \rho_l$ be the distinct moduli of the eigenvalues of A and let $\mathbb{R}^d = E_1^A \oplus \cdots \oplus E_l^A$ be the corresponding invariant splitting into direct sums of the generalized eigenspaces. We denote $A_i = A|E_i^A$. It follows that for any $\epsilon > 0$ there exists C_ϵ such that

$$C_\epsilon^{-1}(\rho_i - \epsilon)^n \le \|A_i^n u\| \le C_\epsilon(\rho_i + \epsilon)^n \quad \text{for any unit vector } u \in E_i^A,$$

and hence the cocycle $\mathcal{A}_i$ generated by A_i is fiber bunched for any $\beta > 0$. Moreover, any cocycle $\mathcal{B}$ with generator B sufficiently C^0 close to A has the corresponding invariant splitting $\mathbb{R}^d = E_1^B(x) \oplus \cdots \oplus E_l^B(x)$, which is close to that of $\mathcal{A}$ and is β-Hölder for some $\beta > 0$. The corresponding restrictions $\mathcal{B}_i$ satisfy similar estimates and hence are also fiber bunched. Since the conjugacy $C(p)$ maps $E_i^A(p)$ to $E_i^B(p)$, the cocycles $\mathcal{A}_i$ and $\mathcal{B}_i$ have conjugate periodic data. Hence by Theorem 2.4 they are conjugate via a Hölder continuous function C_i and we obtain a conjugacy between $\mathcal{A}$ and $\mathcal{B}$ as the direct sum of C_i. ☐

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