LOCAL RIGIDITY OF LYAPUNOV SPECTRUM FOR TORAL AUTOMORPHISMS

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Abstract. We study the regularity of the conjugacy between an Anosov automorphism L of a torus and its small perturbation. We assume that L has no more than two eigenvalues of the same modulus and that L^4 is irreducible over $\mathbb{Q}$. We consider a volume-preserving C^1-small perturbation f of L. We show that if Lyapunov exponents of f with respect to the volume are the same as Lyapunov exponents of L, then f is $C^{1+\text{Hölder}}$ conjugate to L. Further, we establish a similar result for irreducible partially hyperbolic automorphisms with two-dimensional center bundle.

1. Introduction and Statements of Results

Hyperbolic and partially hyperbolic dynamical systems have been one of the main objects of study in smooth dynamics. Anosov automorphisms of tori are the prime examples of hyperbolic systems. The action of a hyperbolic matrix $L \in SL(d, \mathbb{Z})$ on $\mathbb{R}^d$ induces an automorphism of the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. It is well-known that any diffeomorphism f which is sufficiently C^1 close to L is also Anosov and topologically conjugate to L, i.e. there is a homeomorphism h of $\mathbb{T}^d$ such that

$$L = h^{-1} \circ f \circ h.$$

The conjugacy h is unique in the homotopy class of the identity. It is only Hölder continuous in general, and various sufficient conditions for h to be smooth have been studied. For Anosov systems with one-dimensional stable and unstable distributions coincidence of eigenvalues of the derivatives of return maps at the corresponding periodic points was shown to imply smoothness of h [dlL87, dlLM88, dlL92, P90]. For higher dimensional systems even conjugacy of the derivatives of the corresponding return maps is not sufficient in general [dlL92, dlL02]. Nonetheless, smoothness of the conjugacy was established in higher dimensions under various additional assumptions [dlL02, KS03, dlL04, KS09, G08, GKS11]. The last paper establishes local rigidity for the most general class of hyperbolic automorphisms.

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[GKS11, Theorem 1.1] *Let* $L : \mathbb{T}^d \to \mathbb{T}^d$ *be an irreducible Anosov automorphism such that no three of its eigenvalues have the same modulus. Let* f *be a* C^1*-small perturbation of* L *such that the derivative* $D_p f^n$ *is conjugate to* L^n *whenever* $p = f^n p$*. Then* f *is* $C^{1+Hölder}$ *conjugate to* L*.*

We recall that a toral automorphism L is *irreducible* if it has no rational invariant subspaces, or equivalently if its characteristic polynomial is irreducible over $\mathbb{Q}$. It follows that all eigenvalues of L are simple.

Recently, R. Saghin and J. Yang obtained smoothness of the conjugacy h for a volume preserving perturbation f of an irreducible L if f and L have the same *simple* Lyapunov exponents with respect to the volume [SY]. The main result of this paper is the following theorem, which allows for double Lyapunov exponents and extends Saghin-Yang result to a much broader class of irreducible hyperbolic automorphisms.

Theorem 1.1. *Let* $L : \mathbb{T}^d \to \mathbb{T}^d$ *be an Anosov automorphism such that no three of its eigenvalues have the same modulus and its forth power* L^4 *is irreducible. Let* f *be a volume-preserving* C^2 *diffeomorphism of* $\mathbb{T}^d$ *sufficiently* C^1*-close to* L*. If the Lyapunov exponents of* f *with respect to the volume are the same as the Lyapunov exponents of* L*, then* f *is* $C^{1+Hölder}$ *conjugate to* L*.*

Remark 1.2. *In fact, we need* L *to be irreducible and have no pairs of eigenvalues of the form* $\lambda, -\lambda$ *or* $i\lambda, -i\lambda$*, where* λ *is real. This follows if* L^4 *is irreducible.*

We note that the conjugacy of periodic data assumption in [GKS11, Theorem 1.1] implies equality of Lyapunov exponents with respect to the volume by periodic approximation of Lyapunov exponents [K11, Theorem 1.4]. Examples in [G08] show that irreducibility of L is a necessary in the results of [GKS11], and hence in Theorem 1.1, except when L is conformal on the full stable and unstable distributions.

Remark 1.3. *The following example demonstrates that irreducibility assumption is more crucial in the current setting when compared to periodic data rigidity. Let* $A\colon \mathbb{T}^2 \to \mathbb{T}^2$ *be a hyperbolic automorphism. Then notice that the automorphism* $L\colon (x, y) \mapsto (Ax, Ay)$ *and a perturbation* $L'\colon (x, y) \mapsto (Ax, Ay+\varphi(x))$ *have the same volume Lyapunov spectrum. However, in general,* L *and* L' *are not* C^1 *conjugate, because* L' *may have Jordan blocks in its periodic data. On the other hand, de la Llave established periodic data local rigidity for such* L [dlL02].

Remark 1.4. *The toral automorphisms satisfying the assumptions of Theorem 1.1 are generic in the following sense. Consider the set of matrices in* $SL(d, \mathbb{Z})$ *of norm at most* T*. Then the proportion of matrices corresponding to automorphisms that do not satisfy our assumptions goes to zero as* $T \to \infty$*. Moreover, it can be estimated by* $c\,T^{-\delta}$ *for some* $\delta > 0$*.*

We refer to [GKS11] for the proof of this remark. One just needs to consider the relation which ensures irreducibility of L^4 rather than L.

Now we consider the case of a partially hyperbolic toral automorphism, that is an automorphisms for which *some* of the eigenvalues have modulus different from one. We call a toral automorphism L *totally irreducible* if L^n is irreducible for every $n \in \mathbb{N}$. Such an L has no root of unity as an eigenvalues and hence is ergodic and partially hyperbolic. In fact, [RH05, Lemma A.9] shows that an automorphism L is totally irreducible if and only if L is ergodic with irreducible (over the integers or rationals) characteristic polynomial $\chi_L(t)$ that can not be written as a polynomial of t^n for any $n \geq 2$. Such an L was called a *pseudo-Anosov automorphism* in [RH05].

We consider a totally irreducible automorphism L of $\mathbb{T}^d$ with two-dimensional center bundle, i.e. with exactly two eigenvalues on the unit circle. It was shown by Rodriguez Hertz in [RH05] that such L is stably ergodic, more precisely, for large N any sufficiently C^N-small volume-preserving perturbation of L is also ergodic. The following theorem establishes Lyapunov spectrum rigidity for such toral automorphisms with simple real stable and unstable eigenvalues.

Theorem 1.5. *Let $L : \mathbb{T}^d \to \mathbb{T}^d$ be a totally irreducible automorphism with exactly two eigenvalues of modulus one and simple real eigenvalues away from the unit circle. Let f be a volume-preserving C^N-small perturbation of L such that the Lyapunov exponents of f with respect to the volume are the same as the Lyapunov exponents of L. If $d > 4$ and $N = 5$, or $d = 4$ and $N = 22$, then f is $C^{1+\text{Hölder}}$ conjugate to L. If $d = 4$ and $N = \infty$ then f is C^∞ conjugate to L.*

We note that, in contrast to the hyperbolic case, a small perturbation with different Lyapunov spectrum is not necessarily topologically conjugate to L.

The theorem above should be compared to [SY, Theorem D], where partially hyperbolic automorphisms with one dimensional center are treated. Such automorphisms leave invariant a fibration by circles. Consequently, one can consider the skew product perturbations which have the same volume Lyapunov exponents. In general, such perturbations are not C^1 conjugate the automorphism, in contrast to our Theorem 1.5. The reason for this is that automorphisms with one dimensional center are reducible, i.e. have eigenvalue ± 1.

2. Definitions, notations, and outline of the proof of Theorem 1.1

2.1. **Definitions and notations.** Since f is C^1 close to an Anosov automorphism L, it is also *Anosov*. That is, there exist a splitting of the tangent bundle of $\mathbb{T}^d$ into a direct sum of two Df-invariant continuous distributions $E^{s,f}$ and $E^{u,f}$, a Riemannian metric on $\mathcal{M}$, and numbers ν and $\hat{\nu}$ such that

$$\|D_x f(v^s)\| < \nu < 1 < \hat{\nu} < \|D_x f(v^u)\| \tag{2.1}$$

for any $x \in \mathcal{M}$ and unit vectors $v^s \in E^{s,f}(x)$ and $v^u \in E^{u,f}(x)$. The distributionss $E^{s,f}$ and $E^{u,f}$ are called *stable* and *unstable*. They are tangent to the stable and unstable foliations $W^{s,f}$ and $W^{u,f}$ respectively (see, e.g. [KH95]). The leaves of these foliations are as smooth as f, but in general the distributions $E^{s,f}$ and $E^{u,f}$

are only Hölder continuous transversally to the corresponding foliations. We denote by $E^{s,L}$ and $E^{u,L}$ the stable and unstable distributions of L.

Let $1 < \rho_1 < \cdots < \rho_\ell$ be the distinct moduli of the unstable eigenvalues of L and let $E^{u,L} = E_1^L \oplus E_2^L \oplus \cdots \oplus E_\ell^L$ be the corresponding splitting of the unstable distribution. Since f is C^1-close to L, the unstable distribution $E^{u,f}$ splits into a direct sum of ℓ invariant Hölder continuous distributions close to the corresponding ones for L:

$$E^{u,f} = E_1^f \oplus E_2^f \oplus \cdots \oplus E_\ell^f$$

[Pes04, Section 3.3]. Since no three eigenvalues of L have the same modulus, each E_i^L, and hence E_i^f, is either one- or two-dimensional. The Lyapunov exponent of L on E_i^L is $\chi_i = \log \rho_i$. The assumption that the Lyapunov exponents of f with respect to the invariant volume μ are the same as the Lyapunov exponents of L means that for μ-a.e. x,

$$\lim_{n\to\pm\infty} n^{-1} \log \|Df^n(v)\| = \chi_i = \log \rho_i \quad \text{for all } 0 \neq v \in E_i^f(x).$$

We also consider the distributions $E^f_{(i,j)} = E_i^f \oplus E_{i+1}^f \oplus \ldots \oplus E_j^f$. For any $1 < k \le \ell$, $E^f_{(k,\ell)}$ is a fast part of the unstable distribution and thus it integrates to a Hölder foliation $W^f_{(k,\ell)}$ with smooth leaves [Pes04, Section 3.3].

Notation. We say that an object is C^{1+} if it is C^1 and its differential is Hölder continuous with some positive exponent. We say that a homeomorphism h is C^{1+} along a foliation $\mathcal{F}$ if the restrictions of h to the leaves of $\mathcal{F}$ is C^{1+} and the derivative $Dh|_{\mathcal{F}}$ is Hölder continuous on the manifold.

For any $1 \le k < \ell$, $E^f_{(1,k)}$ is a slow part of the unstable distribution. It also integrates to an f-invariant foliation $W^f_{(1,k)}$ with C^{1+} smooth leaves. This can be seen by viewing L as a partially hyperbolic map with the splitting $E^{s,L} \oplus E^L_{(1,k)} \oplus E^l_{(k+1,\ell)}$. Structural stability of partially hyperbolic systems [HPS77, Theorem 7.1] implies that for a C^1-small perturbation f the "central" foliation survives; that is, $E^f_{(1,k)}$ integrates to a foliation $W^f_{(1,k)}$. See [G08, Lemma 6.1] for an alternative proof in our setup that also gives unique integrability.

Since both weak and strong flags are uniquely integrable and the leaves of the corresponding foliations are at least C^{1+}, for any $1 \le k \le \ell$ the distribution $E_k^f = E^f_{(1,k)} \cap E^f_{(k,\ell)}$ also integrates uniquely to a Hölder foliation

$$V_k^f = W^f_{(1,k)} \cap W^f_{(k,\ell)}$$

with C^{1+} smooth leaves. We use analogous notation for the automorphism L: $E^L_{(i,j)} = E_i^L \oplus \ldots \oplus E_j^L$, and $W^L_{(i,j)}$ and V_i^L are the linear foliations tangent to $E^L_{(i,j)}$ and E_i^L respectively.

2.2. **Outline of the proof of Theorem 1.1.** By the Structural Stability Theorem, there exists a unique bi-Hölder continuous homeomorphism h of $\mathbb{T}^d$ close to the identity in C^0 topology such that

$$h \circ L = f \circ h.$$

First, for any sufficiently C^1-small perturbation of an Anosov automorphism of $\mathbb{T}^d$, the conjugacy h takes the flag of weak foliations for L into the corresponding weak flag for f:

Lemma 2.1 (cf. Lemma 6.3 in [G08] and Lemma 2.1 in [GKS11])**.**
For any $1 \le k \le \ell,\ h(W^L_{(1,k)}) = W^f_{(1,k)}$.

Theorem 1.1 is proved by showing inductively that h is C^{1+} along $W^L_{(1,k)}$ for any k and thus along $W^L_{(1,\ell)} = W^u(L)$. By the same argument, h is C^{1+} along $W^s(L)$ and hence h is C^{1+} by Journé Lemma:

Lemma 2.2 (Journé [J88])**.** *Let $\mathcal{M}_j$ be a manifold and $\mathcal{F}^s_j$, $\mathcal{F}^u_j$ be continuous transverse foliations on $\mathcal{M}_j$ with uniformly smooth leaves, $j = 1, 2$. Suppose that $h : \mathcal{M}_1 \to \mathcal{M}_2$ is a homeomorphism that maps $\mathcal{F}^s_1$ into $\mathcal{F}^s_2$ and $\mathcal{F}^u_1$ into $\mathcal{F}^u_2$. Moreover, assume that the restrictions of h to the leaves of these foliations are uniformly $C^{r+\alpha}$, $r \in \mathbb{N}$, $0 < \alpha < 1$. Then h is $C^{r+\alpha}$.*

The inductive step is given by the following proposition [GKS11, Proposition 2.4]:

Proposition 2.3. *Suppose that $h(V^L_i) = V^f_i$, $1 \le i \le k-1$, and h is a C^{1+} diffeomorphism along $W^L_{(1,k-1)}$. Then $h(V^L_k) = V^f_k$ and h is a C^{1+} diffeomorphism along $W^L_{(1,k)}$.*

The first part of the proof is to establish that $h(V^L_k) = V^f_k$. This part is identical to the proof in [GKS11]. The second part, that h is a C^{1+} diffeomorphism along $W^L_{(1,k)}$, follows from Journé Lemma and the following proposition:

Proposition 2.4. *If $h(V^L_i) = V^f_i$, then h is a C^{1+} diffeomorphism along V^L_i.*

Since $V^L_1 = W^L_{(1,1)}$, Lemma 2.1 implies that $h(V^L_1) = V^f_1$, and then Proposition 2.4 yields that h is C^{1+} along V^L_1. This provides the base of the induction. Thus to prove Theorem 1.1 it remains to establish Proposition 2.4, which is done in Section 3.

3. Proof of Proposition 2.4

In this section we write

$$\mathcal{V}^L \stackrel{\text{def}}{=} V^L_i, \quad \mathcal{V}^f \stackrel{\text{def}}{=} V^f_i, \quad \mathcal{E}^L \stackrel{\text{def}}{=} E^L_i = T\mathcal{V}^L, \quad \mathcal{E}^f \stackrel{\text{def}}{=} E^f_i = T\mathcal{V}^f.$$

The key ingredients of the proof are establishing conformality of f along $\mathcal{V}^f$ and showing that the Jacobian of f along $\mathcal{V}^f$ is cohomologous to a constant. They require arguments different from those in [GKS11], where periodic data was used in an essential way. After this we show that h is Lipschitz along $\mathcal{V}^L$ as a limit of smooth maps with uniformly bounded derivatives. Then we prove that the measurable derivative of h along $\mathcal{V}^L$ is actually Hölder continuous. We consider the case when $\mathcal{V}^f$ is two-dimensional, as in the one-dimensional case the conformality is trivial and the other arguments are simpler.

3.1. **Conformality of $Df|_{\mathcal{E}^f}$.** Since the linear map L is irreducible, it is diagonalizable over $\mathbb{C}$. Therefore, as the eigenvalues of $L|_{\mathcal{E}^L}$ have the same modulus, $L|_{\mathcal{E}^L}$ is conformal with respect to some norm on $\mathcal{E}^L$.

We view the restriction $F = Df|_{\mathcal{E}^f}$ as a linear cocycle over f, that is an automorphism of the vector bundle $\mathcal{E}^f$ which covers f. We will now show that F is conformal with respect to some Hölder continuous Riemannian norm $\|\cdot\|^f$ on $\mathcal{E}^f$, that is the linear map $F_x = Df|_{\mathcal{E}^f(x)} : \mathcal{E}^f_x \to \mathcal{E}^f_{fx}$ is conformal for every x. Since the bundle $\mathcal{E}^f$ is Hölder continuous, the cocycle F is β-Hölder for some $\beta > 0$. Also, since f is close to L, the cocycle F is close to conformal and therefore *fiber bunched*, that is, for all $x \in \mathbb{T}^d$,

$$\|F_x\| \cdot \|F_x^{-1}\| \cdot \nu^\beta < 1 \quad \text{and} \quad \|F_x\| \cdot \|F_x^{-1}\| \cdot \hat{\nu}^\beta < 1,$$

where $\nu, \hat{\nu}$ are as in (2.1). Also, by the assumption, F has only one Lyapunov exponent with respect to the volume μ. Thus we can apply the following proposition.

Proposition 3.1 (Two-dimensional Continuous Amenable Reduction).
Let f be a $C^{1+Hölder}$ Anosov diffeomorphism of a compact manifold $\mathcal{M}$ preserving a volume μ. Let $\mathcal{E}$ be a vector bundle over $\mathcal{M}$ with two-dimensional fibers, and let $F : \mathcal{E} \to \mathcal{E}$ be a Hölder continuous fiber bunched cocycle over f with one Lyapunov exponent with respect to μ. Then at least one of the following holds:

(1) *F is conformal with respect to a Hölder continuous Riemannian norm on $\mathcal{E}$;*
(2) *F preserves a Hölder continuous one dimensional sub-bundle;*
(3) *F preserves a Hölder continuous field of two transverse lines.*

Proof. We note that (1) is equivalent to preserving a Hölder continuous conformal structure on $\mathcal{E}$.

The proposition essentially follows from the Continuous Amenable Reduction established in [KS13], specifically Theorem 3.4, Corollary 3.8, and the first remark after Theorem 3.4. The conclusion for 2-dimensional fibers yields that F satisfies (1) or (2) or

(3′) There exists a finite cover $\tilde{F} : \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$ of F which preserves a union of $j \geq 2$ distinct at every point Hölder continuous one dimensional sub-bundles $U = E^1 \cup ... \cup E^j$, and the projection of U to $\mathcal{E}$ is invariant under F.

If $j = 2$, then this U is exactly the invariant Hölder continuous field of two lines. We claim that if $j > 2$ then, in fact, (1) holds. Indeed, by passing to another cover $\hat{\mathcal{E}}$ if necessary, we can make E^1 and E^2 orientable and pick nonzero vector fields $v_1(x) \in E^1$ and $v_2(x) \in E^2$. In this basis the lift $\hat{F} : \hat{\mathcal{E}} \to \hat{\mathcal{E}}$ becomes a Hölder continuous diagonal $GL(2,\mathbb{R})$-valued function $A(x) = \text{diag}(a_1(x), a_2(x))$. Then existence of another invariant sub-bundle E^3 implies that the functions a_1 and a_2 are Hölder cohomologous [S13, Lemma 7.1], that is $a_1(x)/a_2(x) = \phi(fx)/\phi(x)$ for some Hölder continuous function ϕ. It follows that A is uniformly quasiconformal, that is $\|A_x^n\| \cdot \|(A_x^n)^{-1}\|$ is uniformly bounded in x and n, where $A_x^n = A(f^{n-1}x)\cdots A(fx)A(x)$. This implies that $\tilde{F}$ and F are also uniformly quasiconformal and hence F is conformal with respect to a Hölder continuous Riemannian norm on $\mathcal{E}$ by [KS10, Corollary 2.5]. □

Now we want to eliminate possibilities (2) and (3) for our cocycle F. Recall that L has two eigenvalues of the same modulus on $\mathcal{E}^L$. Since L^4 is irreducible and thus has simple eigenvalues we conclude that this is a pair of complex conjugate eigenvalues on $\mathcal{E}^L$ different from $i\lambda$, $-i\lambda$. Hence the projective action of $L|_{\mathcal{E}^L}$ on the lines through 0 does not have a fixed point or an invariant two-point set. This property persists for any sufficiently close linear map, and thus holds for $F_p = Df|_{\mathcal{E}^f(p)}$ at the fixed point $p = f(p)$ close to 0. Hence F can not satisfy (2) or (3) and thus must satisfy (1).

3.2. **Jacobian along $\mathcal{V}^f$.** In this section we show that the Jacobian of f along $\mathcal{V}^f$ is cohomologous to a constant. This result can be deduced using [SY, Corollary G]. Our argument is based on the same idea, but adapted to our setting becomes much shorter, so we include it for reader's convenience.

Denote by m the standard Lebesgue measure on $\mathbb{T}^d$, which is preserved by L, and by μ the smooth f-invariant measure. Then $\mu = h_*(m)$ since both coincide with the unique measure of maximal entropy for f. Indeed, $h_*(m)$ has maximal entropy since m has maximal entropy for L, and μ has the same entropy since by the Pesin formula it equals the sum of positive Lyapunov exponents for f, which are the same as for L.

By the assumption of the proposition we have $h(\mathcal{V}^L) = \mathcal{V}^f$. While $\mathcal{V}^f$, a priori, is not necessarily an absolutely continuous foliation, we claim that the conditional measures of μ on the leaves of $\mathcal{V}^f$ are absolutely continuous. This follows from the assumption on the Lyapunov exponents and Lemma 3.2 below.

Let $\mathcal{V}$ be an expanding foliation for f. We say that a measurable partition ξ is subordinate to $\mathcal{V}$ if it satisfies the following properties.

(1) For all x, $\xi(x) \subset \mathcal{V}(x)$ and for a.e. x partition element $\xi(x)$ is bounded and contains a neighborhood of x in $\mathcal{V}(x)$;
(2) $\vee_{i\geq 0} f^{-i}\xi$ is the partition into points;
(3) $f\xi < \xi$.

Lemma 3.2. *(Ledrappier* [L84], [SY, Lemma 5.5]*) Let $\mathcal{V}$ be an expanding foliation for f, let μ be an invariant measure, and let ξ be a measurable partition subordinate* *to $\mathcal{V}$. Then the conditional measures of μ are absolutely continuous on leaves of $\mathcal{V}$ if and only if the following formula for conditional entropy holds*

$$H_\mu(f^{-1}\xi|\xi) = \int \log Jac\,(f|_{\mathcal{V}})\, d\mu.$$

We consider a measurable partition ξ subordinate to $\mathcal{V} = \mathcal{V}^f$, which exists for example by [LS83]. We let $\xi' = h^{-1}\xi$. Since $h_*m = \mu$, applying the lemma to $L, \mathcal{V}^L, \xi'$ and m we get

$$H_\mu(f^{-1}\xi|\xi) = H_m(L^{-1}\xi'|\xi') = \int \log \mathrm{Jac}(L|_{\mathcal{V}^L})\, dm = \int \log \mathrm{Jac}(f|_{\mathcal{V}^f})\, d\mu.$$

The last equality holds since the integrals are equal to the sums of the Lyapunov exponents of L and f corresponding to the foliations $\mathcal{V}^L$ and $\mathcal{V}^f$, which are the same by the assumption. Applying the lemma to $f, \mathcal{V}^f, \xi$, and μ we conclude that the conditional measures of μ on the leaves of $\mathcal{V}^f$ are absolutely continuous.

We denote by $\{\mu_x = \mu_{\mathcal{V}^f(x)} : x \in \mathbb{T}^d\}$ the system of conditional measures for μ on the leaves of $\mathcal{V}^f$ obtained by pushing forward by h the standard volume on $\mathcal{V}^L$. It is defined for all $x \in \mathbb{T}^d$, satisfies $\mu_y = \mu_x$ for all $y \in \mathcal{V}^f(x)$, and is pushed by f as

$$f_*(\mu_x) = k\,\mu_{fx}, \quad \text{where } k^{-1} = \det L|_{\mathcal{V}^L}.$$

Let σ_x the volume on $\mathcal{V}^f(x)$ induced by the standard metric on $\mathbb{T}^d$. Then absolute continuity of the conditional measures of μ yields that there exists a measurable function ρ on $\mathbb{T}^d$ such that $\mu_x = \rho\,\sigma_x$ for μ-a.e. x. This function is given by the Radon-Nikodym derivative

$$\rho(y) = \frac{d\mu_x}{d\sigma_x}(y) = \lim_{r\to 0} \frac{\mu_x(B^{\mathcal{V}^f}(y,r))}{\sigma_x(B^{\mathcal{V}^f}(y,r))} \quad \text{for a.e. } y \in \mathcal{V}^f(x).$$

We denote the Jacobian of f with respect to the volumes σ_x and σ_{fx} by

$$J(x) = \mathrm{Jac}(f|_{\mathcal{V}^f})(x) = \frac{d\sigma_{fx}}{d(f_*\sigma_x)}(x).$$

The function $J(x)$ is Hölder continuous on $\mathbb{T}^d$ and satisfies $f_*(\sigma_x) = J^{-1}\,\sigma_{fx}$. Then $f_*(\mu_x) = k\,\mu_{fx}$ yields that $\rho(x) = kJ(x)\rho(fx)$ for μ-a.e. x. Indeed,

$$(\rho\circ f^{-1})f_*(\sigma_x) = f_*(\rho\sigma_x) = f_*(\mu_x) = k\,\mu_{fx} = k\rho\,\sigma_{fx} = (J\circ f^{-1})k\rho f_*(\sigma_x).$$

We also note that $\rho(x) > 0$ for μ-a.e. x since μ is equivalent to the standard volume. Thus we have

$$\det L|_{\mathcal{V}^L} = J(x)\rho(fx)\rho(x)^{-1} \quad \text{for } \mu\text{-a.e. } x,$$

that is, J is measurably cohomologous to the constant $\det L|_{\mathcal{V}^L}$. By the measurable Livšic theorem for scalar cocycles [Liv71], ρ coincides μ-a.e. with a Hölder continuous function on $\mathbb{T}^d$.

3.3. **The conjugacy h is Lipschitz along $\mathcal{V}^f$.** The proof is an adaptation of arguments in [dlL02, GKS11]. Let $\bar{\mathcal{V}}^L$ be the linear integral foliation of

$$E^{s,L} \oplus E_1^L \oplus \ldots \oplus E_{i-1}^L \oplus E_{i+1}^L \oplus \ldots \oplus E_l^L.$$

We define a map h_0 by intersecting local leaves:

$$h_0(x) = \mathcal{V}^{f,loc}(h(x)) \cap \bar{\mathcal{V}}^{L,loc}(x).$$

It is easy to check (see [GKS11]) that h_0 is well-defined, close to h, and satisfies

(1) $h_0(\mathcal{V}^L) = \mathcal{V}^f$, moreover, $h_0(\mathcal{V}^L(x)) = \mathcal{V}^f(h(x))$ for all x in $\mathbb{T}^d$;
(2) $\sup_{x\in\mathbb{T}^d} d_{\mathcal{V}^f}(h_0(x), h(x)) < +\infty$, where $d_{\mathcal{V}^f}$ is the distance along the leaves;
(3) h_0 is C^{1+} diffeomorphism along the leaves of $\mathcal{V}^L$;
(4) $h = \lim_{n\to\infty} h_n$ uniformly on $\mathbb{T}^d$, where $h_n = f^{-n} \circ h_0 \circ L^n$.

To prove that h is Lipschitz along $\mathcal{V}^f$ we show that the derivatives of the maps h_n along $\mathcal{V}^L$ are uniformly bounded. With the notation $F_x^n = Df^n|_{\mathcal{E}^f(x)}$ we obtain

$$\begin{aligned}\|D_{\mathcal{V}^L(x)} h_n\| &\le \|(F^n_{f^{-n}(h_0(L^n x)))})^{-1}\| \cdot \|D_{\mathcal{V}^L(L^n x)} h_0\| \cdot \|L^n|_{\mathcal{E}^L}\| \\ &\le \|(F^n_{h_n(x))})^{-1}\| \cdot \|L^n|_{\mathcal{E}^L}\| \cdot \sup\{\|D_{\mathcal{V}^L(z)} h_0\| : z \in \mathbb{T}^d\}.\end{aligned}$$

Since $D_{\mathcal{V}^L} h_0$ is continuous on $\mathbb{T}^d$, the last term is finite and it remains to show that $\|(F_x^n)^{-1}\| \cdot \|L^n|_{\mathcal{E}^L}\|$ is uniformly bounded in $x \in \mathbb{T}^d$ and $n \in \mathbb{N}$.

We denote by $\|\cdot\|$ a norm on $\mathcal{E}^L$ for which $L|_{\mathcal{E}^L}$ is conformal, and by $\|\cdot\|_x^f$ a Hölder continuous Riemannian norm on $\mathcal{E}^f$ for which $F = Df|_{\mathcal{E}^f}$ is conformal. Then

$$\|F(v)\|_{f(x)}^f = a(x)\,\|v\|_x^f \;\text{ for any } x \in \mathbb{T}^d,\; v \in \mathcal{E}^f(x), \;\text{ where } a(x) = (\operatorname{Jac} f|_{\mathcal{V}^f(x)})^{\frac{1}{2}}.$$

Hence for the operator norm with respect to $\|\cdot\|^f$ we have

$$\|(F_x)^{-1}\|^f = \left(\|F_x\|^f\right)^{-1} = a(x)^{-1}.$$

Since $\operatorname{Jac} f|_{\mathcal{V}^f}$ is Hölder cohomologous to the constant $\det L|_{\mathcal{E}^L}$, we conclude that $a(x)$ is Hölder cohomologous to the constant $b = \|L|_{\mathcal{E}^L}\|$, i.e. $b/a(x) = \phi(Lx)/\phi(x)$. for some Hölder continuous function $\phi : \mathbb{T}^d \to \mathbb{R}_+$. We conclude that

$$\|L^n|_{\mathcal{E}^L}\| \cdot \|(F_x^n)^{-1}\|^f = b^n (a(f^{n-1}x)\cdots a(fx)a(x))^{-1} = \phi(L^n x)/\phi(x)$$

is uniformly bounded since ϕ is continuous on $\mathbb{T}^d$. Since the norm $\|\cdot\|^f$ is equivalent to $\|\cdot\|$ we obtain that $\|(F_x)^{-1}\| \cdot \|L^n|_{\mathcal{E}^L}\|$ is uniformly bounded in y and n. We conclude that $\|D_{\mathcal{V}^L(x)} h_n\|$ is uniformly bounded in x and n, and hence $h = \lim_{n\to\infty} h_n$ is Lipschitz along $\mathcal{V}^f$.

A similar argument shows that $\|(D_{\mathcal{V}^L(x)} h_n)^{-1}\|$ is uniformly bounded and hence h is bi-Lipschitz along $\mathcal{V}^f$. In particular, $D_{\mathcal{V}^L} h$ exists and is invertible almost everywhere with respect to the volume.

3.4. **The conjugacy h is C^{1+} along $\mathcal{V}^f$.** Differentiating $f \circ h = h \circ L$ along $\mathcal{V}^L$ on a set of full Lebesgue measure we obtain

$$F_{h(x)} \circ D_{\mathcal{V}^L(x)} h = D_{\mathcal{V}^L(Lx)} h \circ L|_{\mathcal{E}^L(x)},$$

i.e., the cocycles $F_{h(x)}$ and $L|_{\mathcal{E}^L(x)}$ are cohomologous with transfer function $D_{\mathcal{V}^L(x)}h$. The bundle $\mathcal{E}^f$ is trivial since it is close to the trivial bundle $\mathcal{E}^L$. Therefore, $F_{h(x)}$ and $L|_{\mathcal{E}^L(x)}$ can be viewed as Hölder continuous $GL(2,\mathbb{R})$-valued cocycles over the automorphism L. While in general measurable transfer functions are not necessarily continuous [PW01, Section 9], for conformal cocycles the measurable transfer function coincides almost everywhere with a Hölder continuous one by [S15, Theorem 2.7]. We conclude that $D_{\mathcal{V}^L(x)}h$ is Hölder continuous, and hence h is a C^{1+} diffeomorphism along $\mathcal{V}^L$. □

4. Proof of Theorem 1.5

Since f is C^1-close to L, it is *partially hyperbolic*, more precisely, there exist a nontrivial Df-invariant splitting $E^s \oplus E^c \oplus E^u$ of the tangent bundle of $\mathbb{T}^d$, a Riemannian metric on $\mathbb{T}^d$, and constants $\nu < 1$, $\hat{\nu} > 1$, γ, $\hat{\gamma}$ such that for any $x \in \mathcal{M}$ and unit vectors $v^s \in E^{s,f}(x)$, $v^c \in E^{c,f}(x)$, and $v^u \in E^{u,f}(x)$,

$$\|D_x f(v^s)\| < \nu < \gamma < \|D_x f(v^c)\| < \hat{\gamma} < \hat{\nu} < \|D_x f(v^u)\|.$$

The sub-bundles $E^{s,f}$, $E^{u,f}$, and $E^{c,f}$ are called, respectively, stable, unstable, and center. The stable and unstable sub-bundles are tangent to the stable and unstable foliations $W^{s,f}$ and $W^{u,f}$, respectively. The leaves of these foliations are as smooth as f. Moreover, as f is a C^1-small perturbation of L, the center bundle $E^{c,f}$ is tangent to a foliation $W^{c,f}$ with C^{1+} leaves.

Since the center Lyapunov exponents of f are the same, we can use the following result by Avila and Viana to conclude that f is not accessible. A partially hyperbolic diffeomorphism f is called *accessible* if any two points in $\mathcal{M}$ can be connected by an *su*-path, that is, by a concatenation of finitely many subpaths which lie entirely in a single leaf of W^s or W^u.

[AV10, Theorem 8.1] *Let L be as in Theorem 1.5. Then there exists a neighborhood U of L in the space of C^N volume preserving diffeomorphisms of $\mathbb{T}^d$ such that if $f \in U$ is accessible then its center Lyapunov exponents are distinct.*

For the perturbation f which is not accessible, it was proved in [RH05, Section 6] (cf. [AV10, Remark 8.3]) that f and L are conjugate by a bi-Hölder homeomorphism h which is $C^{1+\text{Hölder}}$ along the leaves of the center foliation. In the above theorem, the regularity N is the one needed to apply the KAM-type results in [RH05, Section 6], which is exactly the regularity that we require in Theorem 1.5.

Now we claim that h preserves the volume, that is, $h_*(m) = \mu$. The argument is the same as for Anosov case: both $h_*(m)$ and μ are measures of maximal entropy and hence coincide by uniqueness. The uniqueness of the measure of maximal entropy for L is well-known ([B67], see also [Sch16, Theorem 2.6]). The uniqueness for f follows since f and L are topologically conjugate.

Now use an approach similar to the Anosov case, however now the foliations V_i^f are one-dimensional. For each i the conditional measures of μ along V_i^f are absolutely continuous by the argument in Section 3.2. Therefore, their densities are given, up to normalization, by the ratio of the Jacobians [SY, Proposition 2.3 A4]:

$$\frac{d\mu_x}{d\sigma_x}(y) = c \lim_{n\to\infty} \frac{\mathrm{Jac}(f^n|_{\mathcal{V}^f})(y)}{\mathrm{Jac}(f^n|_{\mathcal{V}^f})(x)}.$$

Since $\mathrm{Jac}(f|_{\mathcal{V}^f})(x)$ is Hölder continuous and the foliation $\mathcal{V}^f$ is contracting, it follows that the densities are Hölder continuous on $\mathbb{T}^d$. This implies that Proposition 2.4 holds for each i, that is, if $h(V_i^L) = V_i^f$ then h is a C^{1+} diffeomorphism along V_i^L. Indeed, the conjugacy h maps the conditional measures on the leaves of V_i^L to those on the leaves of V_i^f and hence h^{-1} is obtained by integrating the Hölder continuous density along the one-dimensional leaves.

We note that $h(W^{u,L}) = W^{u,f}$ and $h(W^{s,L}) = W^{s,f}$. Indeed,

$$d(f^n x, f^n y) \to 0 \implies d(L^n h^{-1}(x), L^n h^{-1}(y)) \to 0 \implies h^{-1}(y) \in W^{s,L}(h^{-1}(x))$$

Thus $h^{-1}(W^{s,f}(x)) \subset W^{s,L}(h(x))$ and equality follows as h is a homeomorphism.

Now the same inductive process as in the Anosov case shows that h is a C^{1+} diffeomorphism along $W^{u,L}$, and similarly is C^{1+} along $W^{s,L}$. We recall that h is smooth along the leaves of $W^{c,L}$. Applying Journé Lemma 2.2 twice we obtain that h is C^{1+} along $W^{u,L} \oplus W^{u,L}$ and then on $\mathbb{T}^d$.

If f is C^∞ close to L then [RH05, Section 6] gives that h is also C^∞ along along the center foliation. If $d = 4$ then E^s and E^u are one-dimensional and their leaves are C^∞ manifolds. In this case the densities of the conditional measures are C^∞ along the leaves because they are given by the infinite product of ratios of Jacobians. Then h is C^∞ along these foliations and hence is C^∞ on $\mathbb{T}^d$ by Journé Lemma. ☐

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