MULTIFRACTAL ANALYSIS OF CONFORMAL AXIOM A FLOWS

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Abstract. We develop the multifractal analysis of conformal axiom A flows. This includes the study of the Hausdorff dimension of basic sets of the flow, the description of the dimension spectra for pointwise dimension and for Lyapunov exponents and the multifractal decomposition associated with these spectra. The main tool of study is the thermodynamic formalism for hyperbolic flows by Bowen and Ruelle. Examples include suspensions over axiom A conformal diffeomorphisms, Anosov flows, and in particular, geodesic flows on compact smooth surfaces of negative curvature.

1. Introduction

The multifractal analysis of dynamical systems has recently become a popular topic in the dimension theory of dynamical systems. By now only conformal dynamical systems with discrete time have been subjects of study. They include conformal expanding maps and conformal axiom A diffeomorphisms (see [10] for the definition of conformal axiom A diffeomorphisms, related results, and further references).

In this paper we extend the study to include conformal dynamical systems with continuous time, more precisely, conformal axiom A flows. Our first result is the formula for the Hausdorff dimension of basic sets of axiom A flows (see Section 4). It is an extension to the continuous time case of the famous Bowen pressure formula for the Hausdorff dimension of hyperbolic sets.

We then consider the two dimension spectra: the dimension spectrum for pointwise dimensions generated by Gibbs measures and the dimension spectrum for Lyapunov exponents. Using the symbolic representation of axiom A flows by suspensions over subshifts of finite type and the associated thermodynamic formalism of Bowen and Ruelle ([4]), we obtain a complete description of these spectra. The statements of our results are similar in spirit to those in the discrete time case but proofs require some substantial technical modifications.

We stress that we handle only axiom A flows which are conformal and we introduce and study this notion in Section 3. Examples include suspensions over conformal axiom A diffeomorphisms and two-dimensional Anosov flows. Our results provide, in particular, a formula for the dimension and a description of the dimension spectra for pointwise dimensions and for Lyapunov exponents for the time-one map of the

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flow. This is the first example of a partially hyperbolic diffeomorphism for which such results are now known.

Let us emphasize that, in general, both dimension spectra are non-trivial. More precisely, as we show in Section 5, the dimension spectrum for pointwise dimension is trivial (i.e., is a δ-function) if and only if the Gibbs measure is the measure of full dimension. For an Anosov flow it holds if it preserves a smooth measure.

Furthermore, the dimension spectrum for Lyapunov exponents is trivial if and only if the measure of full dimension coincides with the measure of maximal entropy. We apply this statement to geodesic flows on compact n-dimensional Riemannian manifolds of negative curvature. For $n = 2$ we have that the spectrum is trivial if and only if the topological entropy of the flow coincide with the metric entropy (see Section 3). This provides a new insight into the famous Katok's entropy conjecture (see [8]). For $n > 2$, the requirement that the flow is conformal implies that the curvature is constant. In particular, the dimension spectrum for Lyapunov exponents is trivial.

Finally we describe multifractal decomposition associated with the two spectra. More detailed description can be found in [2].

2. Preliminaries

Let $\mathcal{M}$ be a smooth finite-dimensional Riemannian manifold. Throughout this paper $f^t\colon \mathcal{M} \to \mathcal{M}$ is a flow on $\mathcal{M}$ without fixed points generated by a C^r-vector field V, $r \geq 1$, i.e., $\frac{df^t(x)}{dt} = V(x)$ for every $x \in \mathcal{M}$. A compact f^t-invariant set $\Lambda \subset \mathcal{M}$ is said to be **hyperbolic** if there exist a continuous splitting of the tangent bundle $T_\Lambda \mathcal{M} = E^{(s)} \oplus E^{(u)} \oplus X$ and constants $C > 0$ and $0 < \lambda$ such that for every $x \in \Lambda$ and $t \in \mathbb{R}$,

1. $df^t\left(E^{(s)}(x)\right) = E^{(s)}(f^t(x))$, $df^t\left(E^{(u)}(x)\right) = E^{(u)}(f^t(x))$, and $X = \{\alpha V : \alpha \in \mathbb{R}\}$ is a one-dimensional subbundle;
2. for all $t \geq 0$,
$$\|df^t v\| \leq Ce^{-\lambda t}\|v\| \quad \text{if } v \in E^{(s)}(x),$$
$$\|df^{-t} v\| \leq Ce^{-\lambda t}\|v\| \quad \text{if } v \in E^{(u)}(x).$$

The subspaces $E^{(s)}(x)$ and $E^{(u)}(x)$ are called **stable** and **unstable subspaces** at x respectively and they depend Hölder continuously on x. It is well-known (see, for example, [8]) that for every $x \in \Lambda$ one can construct **stable** and **unstable local manifolds**, $W^{(s)}_{\rm loc}(x)$ and $W^{(u)}_{\rm loc}(x)$. They have the following properties:

3. $x \in W^{(s)}_{\rm loc}(x)$, $x \in W^{(u)}_{\rm loc}(x)$;
4. $T_x W^{(s)}_{\rm loc}(x) = E^{(s)}(x)$, $T_x W^{(u)}_{\rm loc}(x) = E^{(u)}(x)$;
5. $f^t(W^{(s)}_{\rm loc}(x)) \subset W^{(s)}_{\rm loc}(f^t(x))$, $f^{-t}(W^{(u)}_{\rm loc}(x)) \subset W^{(u)}_{\rm loc}(f^{-t}(x))$;
6. there exist $K > 0$ and $0 < \mu$ such that for every $t \geq 0$,
$$\rho(f^t(y), f^t(x)) \leq Ke^{-\mu t}\rho(y, x) \text{ for all } y \in W^{(s)}_{\rm loc}(x)$$

and

$$\rho(f^{-t}(y), f^{-t}(x)) \le Ke^{-\mu t}\rho(y,x) \text{ for all } y \in W^{(u)}_{\text{loc}}(x),$$

where ρ is the distance in $\mathcal{M}$ induced by the Riemannian metric;

A hyperbolic set Λ is called **locally maximal** if there exists a neighborhood U of Λ such that

$$\Lambda = \bigcap_{-\infty<t<\infty} f^t(U).$$

For a locally maximal hyperbolic set Λ the following property holds:

7. for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any two points $x, y \in \Lambda$ with $\rho(x,y) \le \delta$ one can find a number $t = t(x,y)$, $|t| \le \epsilon$, for which the intersection

$$W^{(s)}_{\text{loc}}(f^t(x)) \cap W^{(u)}_{\text{loc}}(y)$$

consists of a single point $z \in \Lambda$. We denote this point by $z = [x,y]$; moreover, the maps $t(x,y)$ and $[x,y]$ are continuous.

We define **stable** and **unstable global manifolds** at $x \in \Lambda$ by

$$W^{(s)}(x) = \bigcup_{t\ge 0} f^{-t}\left(W^{(s)}_{\text{loc}}(f^t(x))\right), \quad W^{(u)}(x) = \bigcup_{t\ge 0} f^{t}\left(W^{(u)}_{\text{loc}}(f^{-t}(x))\right).$$

They can be characterized as follows:

$$W^{(s)}(x) = \{\, y \in \Lambda :\ \rho(f^t(y), f^t(x)) \to 0 \ \text{ as } t \to \infty \,\},$$
$$W^{(u)}(x) = \{\, y \in \Lambda :\ \rho(f^{-t}(y), f^{-t}(x)) \to 0 \ \text{ as } t \to \infty \,\}.$$

A flow f^t is called an **axiom A flow** if its set of non-wandering points is hyperbolic. Let us remark that we deal only with flows without fixed points. If this assumption is dropped one should assume in the above definition that the flow has finitely many hyperbolic fixed points.

The **Smale Spectral Decomposition Theorem** claims (see [8]) that in this case the hyperbolic set can be decomposed into finitely many disjoint closed f^t-invariant locally maximal hyperbolic sets on each of which f^t is topologically transitive. These sets are called **basic sets**.

From now on we will assume that f^t is topologically transitive on a locally maximal hyperbolic set Λ. One can show that periodic orbits are dense in Λ.

In [3], Bowen constructed Markov partitions of basic sets (see also [13] for the case of Anosov flows). We provide here a concise description of his results. Given a point $x \in \Lambda$, consider a small compact disk D containing x of co-dimension one which is transversal to the flow f^t. This disk is a local section of the flow, i.e., there exists $\tau > 0$ such that the map $(y,t) \to f^t(y)$ is a diffeomorphism of the direct product $D \times [-\tau, \tau]$ onto a neighborhood $U_\tau(D)$. The projection $P_D : U_\tau(D) \to D$ is a differentiable map.

Consider now a closed set $\Pi \subset \Lambda \cap D$ which does not intersect the boundary ∂D. For any two points $y, z \in \Pi$ let $\{y,z\} = P_D[y,z]$. The set Π is said to be a

rectangle if $\Pi = \overline{\mathrm{int}\Pi}$ (where the interior of Π is considered with respect to the induced topology of $\Lambda \cap D$) and $\{y, z\} \in \Pi$ for any $y, z \in \Pi$. If Π is a rectangle then for every $x \in \Pi$ we set

$$\begin{aligned} W^{(s)}_{\mathrm{loc}}(x,\Pi) &= \{\, \{x,y\} : y \in \Pi \,\} = \Pi \cap P_D \left(U_\tau(D) \cap W^{(s)}_{\mathrm{loc}}(x) \right), \\ W^{(u)}_{\mathrm{loc}}(x,\Pi) &= \{\, \{z,x\} : z \in \Pi \,\} = \Pi \cap P_D \left(U_\tau(D) \cap W^{(u)}_{\mathrm{loc}}(x) \right) \end{aligned} \tag{2.1}$$

(we assume that $\operatorname{diam}\Pi$ is much smaller than the size of local stable and unstable manifolds).

A collection of rectangles $\mathcal{T} = \{\Pi_1, \dots, \Pi_n\}$ is called **regular of size** r_0 if there exist small compact co-dimension one disks $D_1, \dots, D_n$, which are transversal to the flow f^t, such that

1. $\mathrm{diam} D_i < r_0$ and $\Pi_i \subset \mathrm{int} D_i$;
2. for $i \neq j$ at least one of the sets $D_i \cap f^{[0,r_0]} D_j$ or $D_j \cap f^{[0,r_0]} D_i$ is empty; in particular, $D_i \cap D_j = \emptyset$;
3. $\Lambda = f^{[-r_0,0]} \Gamma(\mathcal{T})$ where $\Gamma(\mathcal{T}) = \Pi_1 \cup \cdots \cup \Pi_n$.

Let $\mathcal{T} = \{\Pi_1, \dots, \Pi_n\}$ be a regular collection of rectangles of size r_0. For every $x \in \Gamma(\mathcal{T})$ one can find the smallest positive number $t(x) \le r_0$ such that $f^{t(x)}(x) \in \Gamma(\mathcal{T})$. Since the disks D_i are disjoint there exists a number $\beta > 0$ such that $t(x) \ge \beta$ for all x. The map $H_{\mathcal{T}} : \Gamma(\mathcal{T}) \to \Gamma(\mathcal{T})$ given by

$$H_{\mathcal{T}}(x) = f^{t(x)}(x) \tag{2.2}$$

is one-to-one. Note that the maps $t(x)$ and $H_{\mathcal{T}}$ are not continuous on $\Gamma(\mathcal{T})$ but on

$$\Gamma'(\mathcal{T}) = \left\{ x \in \Gamma(\mathcal{T}) : (H_{\mathcal{T}})^k(x) \in \bigcup_{i=1}^{n} \mathrm{int}\, \Pi_i \ \text{ for all } \ k \in \mathbb{Z} \right\}. \tag{2.3}$$

The set $\Gamma'(\mathcal{T})$ is dense in $\Gamma(\mathcal{T})$ and the set $\cup_{t \in \mathbb{R}} f^t(\Gamma'(\mathcal{T}))$ is dense in Λ.

Given two rectangles Π_i and Π_j we denote by

$$\begin{aligned} U(\Pi_i, \Pi_j) &= \overline{\{w \in \Gamma'(\mathcal{T}) :\ w \in \Pi_i,\ H_{\mathcal{T}}(w) \in \Pi_j\}}, \\ V(\Pi_i, \Pi_j) &= \overline{\{w \in \Gamma'(\mathcal{T}) :\ w \in \Pi_j,\ H_{\mathcal{T}}^{-1}(w) \in \Pi_i\}}. \end{aligned} \tag{2.4}$$

A **Markov collection of size** r_0 (for a basic set Λ) is a regular collection $\mathcal{T} = \{\Pi_1, \dots, \Pi_n\}$ of rectangles of size r_0 which satisfies the following conditions: for any $1 \le i, j \le n$,

1. if $x \in U(\Pi_i, \Pi_j)$ then $W^{(s)}_{\mathrm{loc}}(x, \Pi) \subset U(\Pi_i, \Pi_j)$;
2. if $y \in V(\Pi_i, \Pi_j)$ then $W^{(u)}_{\mathrm{loc}}(y, \Pi) \subset V(\Pi_i, \Pi_j)$

(see (2.1)). In [3], Bowen proved that for any sufficiently small r_0 there exist a Markov collection of size r_0.

Given a rectangle $\Pi_i \in \mathcal{T}$, we call the set

$$R_i = \bigcup_{x\in\Pi_i} \bigcup_{0\le t\le t(x)} f^t(x) \subset \Lambda \tag{2.5}$$

a **Markov set** (corresponding to the Markov collection $\mathcal{T}$). Note that $R_i = \overline{\mathrm{int}R_i}$ and $\mathrm{int}R_i \cap \mathrm{int}R_j = \emptyset$ for any $i \neq j$.

Using Markov collections one can obtain symbolic representations of Axiom A flows by symbolic suspension flows (see Appendix; see also [4]).

Proposition 2.1. *Let Λ be a basic set for an axiom A flow f^t generated by a C^1-vector field V. Then there exists a topologically mixing subshift of finite type (Σ_A, σ) (see Appendix), a positive Hölder continuous function ψ (in the metric d_β for some $\beta > 1$, (see (A.22)), and a continuous projection map $\chi : \Lambda(A, \psi) \to \Lambda$ such that the following diagram*

$$\begin{array}{ccc}
\Lambda(A,\psi) & \xrightarrow{S^t} & \Lambda(A,\psi) \\
\chi\downarrow & & \downarrow\chi \\
\Lambda & \xrightarrow{f^t} & \Lambda
\end{array}$$

is commutative with S^t a symbolic suspension flow (see (A.24)).

The map χ is called the **coding map**.

The transfer matrix $A = (a_{i,j})$ is uniquely determined by a Markov collection for Λ. Namely, if $\mathcal{T} = \{\Pi_1, \dots, \Pi_n\}$ is such a collection then $a_{i,j} = 1$ if and only if there exists a point $x \in \Gamma'(\mathcal{T})$ such that $x \in \Pi_i$ and $H_{\mathcal{T}}(x) \in \Pi_j$ (see (2.2) and (2.3)).

As an immediate consequence of Proposition 2.1 we obtain the following statement.

Proposition 2.2. *Let Λ be a basic set for an axiom A flow f^t and $\varphi : \Lambda \to \mathbb{R}$ a Hölder continuous function. Then there exists a unique equilibrium measure ν_φ corresponding to φ (see (A.21)). Moreover, the measure ν_φ is ergodic and positive on open sets.*

We describe the local structure of an equilibrium measure ν corresponding to a Hölder continuous function (see part 3 of the Appendix).

Let $R_1, \dots, R_n$ be the Markov sets corresponding to a Markov collection $\mathcal{T}$ for Λ. Let us fix a set R_i and consider the partitions $\xi^{(u)}$ and $\xi^{(s)}$ of R_i by local stable and unstable manifolds. Denote by $\nu^{(u)}(x)$ and $\nu^{(s)}(x)$ the corresponding conditional measures on $W^{(u)}_{\mathrm{loc}}(x) \cap R_i$ and $W^{(s)}_{\mathrm{loc}}(x) \cap R_i$ (where $x \in R_i$) generated by ν. The following statement shows that equilibrium measures have local product structure. Its proof follows from Proposition 8.5 (see Appendix) and local product structure of Gibbs measures for subshifts of finite type (see [10]).

Proposition 2.3. *There are positive constants A_1 and A_2 such that for some point $x \in R_i$ and any Borel set $E \subset R_i$*

$$A_1 \int_{R_i} \chi_E(y,z,t)\, d\nu^{(u)}(y)\, d\nu^{(s)}(z)\, dt \;\le\; \nu(E) \;\le\; A_2 \int_{R_i} \chi_E(y,z,t)\, d\nu^{(u)}(y)\, d\nu^{(s)}(z)\, dt, \tag{2.6}$$

where $y \in W^{(u)}_{loc}(x)$ and $z \in W^{(s)}_{loc}(x)$.

3. Conformal Axiom A Flows

Let $F = \{f^t\}$ be a C^2-flow on a locally maximal hyperbolic set Λ. We say that F is u-**conformal** (respectively, s-**conformal**) if there exists a continuous function $A^{(u)}$ (respectively, $A^{(s)}$) on $\Lambda \times \mathbb{R}$ such that for every $x \in \Lambda$ and $t \in \mathbb{R}$,

$$df^t|_{E^{(u)}(x)} = A^{(u)}(x,t)\, I^{(u)}(x,t),$$

respectively,

$$df^t|_{E^{(s)}(x)} = A^{(s)}(x,t)\, I^{(s)}(x,t),$$

where $I^{(u)}(x,t) : E^{(u)}(x) \to E^{(u)}(f^t x)$ and $I^{(s)}(x,t) : E^{(s)}(x) \to E^{(s)}(f^t x)$ are isometries.

We define functions $a^{(u)}(x)$ and $a^{(s)}(x)$ by

$$a^{(u)}(x) \;=\; \frac{\partial}{\partial t} \log A^{(u)}(x,t)\,|_{t=0} \;=\; \lim_{t\to 0} \frac{\log \|\, df^t|_{E^{(u)}(x)} \,\|}{t},$$

$$a^{(s)}(x) \;=\; \frac{\partial}{\partial t} \log A^{(s)}(x,t)\,|_{t=0} \;=\; \lim_{t\to 0} \frac{\log \|\, df^t|_{E^{(s)}(x)} \,\|}{t}.$$

Since the subspaces $E^{(u)}(x)$ and $E^{(s)}(x)$ depend Hölder continuously on x the functions $a^{(u)}(x)$ and $a^{(s)}(x)$ are also Hölder continuous. Note that $a^{(u)}(x) > 0$ and $a^{(s)}(x) < 0$ for every $x \in \Lambda$. For any $x \in \Lambda$ and any $t \in \mathbb{R}$, we have

$$\|\, df^t(v) \,\| \;=\; \|v\| \,\exp \int_0^t a^{(u)}(f^\tau(x))\, d\tau \quad \text{for any } \; v \in E^{(u)}(x), \tag{3.1}$$

and

$$\|\, df^t(w) \,\| \;=\; \|w\| \,\exp \int_0^t a^{(s)}(f^\tau(x))\, d\tau \quad \text{for any } \; w \in E^{(s)}(x). \tag{3.2}$$

A flow $F = \{f^t\}$ on Λ is called **conformal** if it is u-conformal and s-conformal as well. It is easy to see that a three-dimensional flow on a locally maximal hyperbolic set is conformal.

If $F = \{f^t\}$ is a conformal flow then for every $x \in \Lambda$ the Lyapunov exponent at x takes on two values which are given by

$$\lambda^+(x) = \lim_{t\to\infty} \frac{\log \|df^t_x|_{E^{(u)}(x)}\|}{t} = \lim_{t\to\infty} \frac{1}{t} \int_0^t a^{(u)}(f^\tau(x))\, d\tau > 0, \tag{3.3}$$

$$\lambda^-(x) = \lim_{t\to\infty} \frac{\log \|df_x^t|_{E^{(s)}(x)}\|}{t} = \lim_{t\to\infty} \frac{1}{t}\int_0^t a^{(s)}(f^\tau(x))\,d\tau < 0 \tag{3.4}$$

(provided the limit exists). If ν is an f-invariant measure then by the Birkhoff ergodic theorem, the above limits exist ν-almost everywhere, and if ν is ergodic then they are constant almost everywhere. We denote the corresponding values by $\lambda_\nu^+ > 0$ and $\lambda_\nu^- < 0$.

We describe some examples of conformal axiom A flows.

1. A suspension flow over a conformal axiom A diffeomorphism is a conformal axiom A flow. Note that if the height function of a suspension flow is not cohomologous to a constant then the corresponding suspension flow is mixing.

2. Consider a conformal Anosov flow F. Let Λ be a closed locally maximal hyperbolic set for F. Then the restriction of $F|_\Lambda$ is a conformal axiom A flow.

3. Consider the geodesic flow on a compact Riemannian manifold $\mathcal{M}$ of negative curvature. The flow acts on the space $S\mathcal{M} = \{(x,v) : x \in \mathcal{M},\ v \in T_x\mathcal{M},\ \|v\| = 1\}$ of unit tangent vectors. We endow the second tangent bundle $TT\mathcal{M}$ with a Riemannian metric whose projection to $T\mathcal{M}$ is the given metric. If $\dim \mathcal{M} = 2$ then the geodesic flow is conformal since stable and unstable subspaces are one dimensional, and our results apply.

If $\dim \mathcal{M} \geq 3$ the result in [7] shows that conformality of the geodesic flow implies that $\mathcal{M}$ is of constant curvature (regardless of the metric on the second tangent bundle). We thank M. Kanai for imforming us on his result.

On the other hand, if the curvature of $\mathcal{M}$ is constant then the geodesic flow is conformal provided the second tangent bundle is endowed with the canonical metric.

Remark 3.1. Our main results (Theorems 4.1, 4.2, 5.1, 5.2, and 5.3) can be easily generalized to the case when the flow is not conformal, but has bounded distortion. By this we mean that there exist Hölder continuous functions $a^{(u)}$ and $a^{(s)}$ on Λ, and constants $K_1, K_2 > 0$ such that for any $x \in \Lambda$, $v \in E^{(u)}(x)$, $w \in E^{(s)}(x)$, and $t \in \mathbb{R}$,

$$K_1\,\|v\|\,\exp\int_0^t a^{(u)}(f^\tau(x))\,d\tau \;\leq\; \|\,df^t(v)\,\| \;\leq\; K_2\,\|v\|\,\exp\int_0^t a^{(u)}(f^\tau(x))\,d\tau,$$

and

$$K_1\,\|w\|\,\exp\int_0^t a^{(s)}(f^\tau(x))\,d\tau \;\leq\; \|\,df^t(w)\,\| \;\leq\; K_2\,\|w\|\,\exp\int_0^t a^{(s)}(f^\tau(x))\,d\tau$$

(compare to (3.1) and (3.2)). We thank A. Katok for providing us with this remark.

4. Hausdorff and Box Dimension of Basic Sets for Conformal Axiom A Flows

Let Λ be a basic set for a u-conformal axiom A flow $F = \{f^t\}$. Consider the function

$$-t^{(u)}a^{(u)}(x) \tag{4.1}$$

on Λ, where $t^{(u)}$ is a unique root of Bowen's equation

$$P_\Lambda(F, -t\ a^{(u)}) = 0 \tag{4.2}$$

(see (A.16)-(A.18)). The function $-t^{(u)}a^{(u)}$ is Hölder continuous and therefore, there exists a unique equilibrium measure corresponding to it. We denote this measure by $\kappa^{(u)}$.

Let $\mathfrak{T} = \{\Pi_1, \dots, \Pi_n\}$ be a Markov collection for Λ and $R_1, \dots, R_n$ the Markov sets corresponding $\mathfrak{T}$. Given $x \in \Lambda$ denote by $R(x)$ a Markov set containing x. Consider the conditional measures $\eta^{(u)}(y)$ on $W^{(u)}_{\text{loc}}(y) \cap R(x)$ (where $y \in R(x)$) generated by the measure $\kappa^{(u)}$.

We now state the result which describes the Hausdorff dimension of subsets of unstable manifolds.

Theorem 4.1. *For any $x \in \Lambda$ and any open set $U \subset W^{(u)}_{loc}(x)$ such that $U \cap \Lambda \neq \emptyset$ the following statements hold:*

(1) $\dim_H(U \cap \Lambda) = \underline{\dim}_B(U \cap \Lambda) = \overline{\dim}_B(U \cap \Lambda) = t^{(u)}$;

(2)
$$t^{(u)} = \frac{h_{\kappa^{(u)}}(f^1)}{\int_\Lambda a^{(u)}(y)\, d\kappa^{(u)}(y)}, \tag{4.3}$$
where $h_{\kappa^{(u)}}(f^1)$ is the measure-theoretic entropy of the time-one map f^1 with respect to the measure $\kappa^{(u)}$;

(3) $d_{\eta^{(u)}(x)}(y) = t^{(u)}$ *for all* $y \in W^{(u)}_{loc}(x) \cap R(x)$;

(4) $t^{(u)} = \dim_H \eta^{(u)}(x)$, *i.e., the measure $\eta^{(u)}(x)$ is the measure of full dimension (see Appendix);*

(5) *the $t^{(u)}$-Hausdorff measure of $U \cap \Lambda$ is positive and finite; moreover, it is equivalent to the measure $\eta^{(u)}(x)|_U$.*

Remark 4.1 Consider a u-conformal diffeomorphism f on a locally maximal hyperbolic set X. This means that there exists a continuous function $b^{(u)}$ on X such that for any $x \in X$

$$df|_{E^{(u)}(x)} = b^{(u)}(x)\, I^{(u)}(x),$$

where $I^{(u)}(x) : E^{(u)}(x) \to E^{(u)}(f(x))$ is an isometry (see [10]). It is known that for any $x \in X$ and any open set $U \subset W^{(u)}_{\text{loc}}(x)$ such that $U \cap X \neq \emptyset$,

$$\dim_H(U \cap X) = \underline{\dim}_B(U \cap X) = \overline{\dim}_B(U \cap X) = t^{(u)},$$

where $t^{(u)}$ is the unique root of Bowen's equation

$$P_X(f, -t \log b^{(u)}) = 0$$

(see [10]).

Consider a u-conformal flow $F = \{f^t\}$ and the corresponding time-one map f^1. It is a partially hyperbolic diffeomorphism and the local strong unstable manifold for f^1 at a point $x \in \Lambda$, $W^{(su)}_{\text{loc}}(x)$, coincides with $W^{(u)}_{\text{loc}}(x)$ for the flow F. Note that

$$df^1|_{E^{(su)}(x)} = A^{(u)}(x,1)\, I^{(u)}(x,1).$$

In view of (A.19)

$$P_\Lambda(F, -ta^{(u)}) = P_\Lambda\left(f^1, -t\int_0^1 a^{(u)}(f^\tau x)\, d\tau\right) = P_\Lambda\left(f^1, -t\log A^{(u)}(x,1)\right).$$

Therefore, the first statement of Theorem 4.1 and (4.2) imply that for any $x \in \Lambda$ and for any open set $U \subset W^{(su)}_{\text{loc}}(x)$ such that $U \cap \Lambda \neq \emptyset$

$$\dim_H(U\cap\Lambda) = \underline{\dim}_B(U\cap\Lambda) = \overline{\dim}_B(U\cap\Lambda) = t^{(u)},$$

where $t^{(u)}$ is the unique root of Bowen's equation

$$P_\Lambda(f^1, -t\log A^{(u)}(x,1)) = 0.$$

This gives a formula for the dimension of $W^{(su)}_{\text{loc}}(x) \cap \Lambda$ for the partially hyperbolic time-one diffeomorphism f^1. This formula is the same as the one for a u-conformal diffeomorphism.

It is not known in general how to compute the dimension of $W^{(su)}_{\text{loc}}(x) \cap \Lambda$ for an arbitrary partially hyperbolic diffeomorphism.

We now consider a basic set Λ for an s-conformal axiom A flow $F = \{f^t\}$. Similarly to (4.1) and (4.2) define the function

$$t^{(s)} a^{(s)}(x) \tag{4.4}$$

on Λ where $t^{(s)}$ is a unique root of Bowen's equation

$$P_\Lambda(F, ta^{(s)}) = 0 \tag{4.5}$$

(see (A.16)-(A.18)). The function $t^{(s)}a^{(s)}$ is Hölder continuous and therefore, there exists a unique equilibrium measure corresponding to it. We denote this measure by $\kappa^{(s)}$.

Given $x \in \Lambda$ consider the conditional measures $\eta^{(s)}(y)$ on $W^{(s)}_{\text{loc}}(y) \cap R(x)$ (where $y \in R(x)$) generated by the measure $\kappa^{(s)}$ on a Markov set $R(x)$ containing x.

Similarly to Theorem 4.1, one can prove that for any $x \in \Lambda$ and any open set $U \subset W^{(s)}_{\text{loc}}(x)$,

$$\dim_H(U\cap\Lambda) = \underline{\dim}_B(U\cap\Lambda) = \overline{\dim}_B(U\cap\Lambda) = t^{(s)}.$$

Moreover,

$$t^{(s)} = -\frac{h_{\kappa^{(s)}}(f^1)}{\int_\Lambda a^{(s)}(y)\, d\kappa^{(s)}(y)}, \tag{4.6}$$

where $h_{\kappa^{(s)}}(f^1)$ is the measure-theoretic entropy of the time-one map f^1 with respect to the measure $\kappa^{(s)}$.

The $t^{(s)}$-Hausdorff measure of $U \cap \Lambda$ is positive and finite. In addition, $d_{\eta^{(s)}(x)}(y) = t^{(s)}$ for all $y \in W^{(s)}_{\text{loc}}(x) \cap R(x)$, and therefore $\dim_H \eta^{(s)}(x) = t^{(s)}$, i.e. the measure $\eta^{(s)}(x)$ is the measure of full dimension.

We now consider the case when Λ is a basic set for an axiom A flow $F = \{f^t\}$ which is both s- and u-conformal. Using Proposition 7.1 we compute the Hausdorff dimension and box dimension of Λ.

Theorem 4.2. *We have*

$$\dim_H \Lambda = \underline{\dim}_B \Lambda = \overline{\dim}_B \Lambda = t^{(u)} + t^{(s)} + 1,$$

where $t^{(u)}$ and $t^{(s)}$ are unique roots of Bowen's equations (4.2) and (4.5) and can be computed by the formulae (4.3) and (4.6).

This result applies and produces a formula for the Hausdorff dimension and box dimension of a basic set of an Axiom A flow on a surface which is clearly seen to be both s- and u-conformal.

Consider the measures $\kappa^{(u)}$ and $\kappa^{(s)}$ on Λ, which are equilibrium measures for the functions $-t^{(u)}a^{(u)}$ and $t^{(s)}a^{(s)}$ respectively. It is easy to see that

$$\dim_H \kappa^{(u)} \le t^{(u)} + t^{(s)} + 1, \quad \dim_H \kappa^{(s)} \le t^{(u)} + t^{(s)} + 1.$$

Moreover, the equalities hold if and only if

$$\kappa^{(u)} = \kappa^{(s)} \stackrel{\text{def}}{=} \kappa. \tag{4.7}$$

In this case, κ is the measure of full dimension. Condition (4.7) is a "rigidity" type condition. It holds if and only if the functions $-t^{(u)}a^{(u)}(x)$ and $t^{(s)}a^{(s)}(x)$ are cohomologous (see [8]). One can show that this is the case if and only if for any periodic point $x \in \Lambda$ of period p,

$$t^{(u)} \int_0^p a^{(u)}(f^\tau(x))\, d\tau = -t^{(s)} \int_0^p a^{(s)}(f^\tau(x))\, d\tau.$$

5. Multifractal Analysis of Conformal Axiom A Flows on Basic Sets

We undertake the complete multifractal analysis of equilibrium measures on a locally maximal hyperbolic set Λ of a flow $F = \{f^t\}$ assuming that the flow is both s- and u-conformal. We follow the approach suggested by Pesin and Weiss in [11] (see also [10]).

Let φ be a Hölder continuous function on Λ and $\nu = \nu_\varphi$ a unique equilibrium measure for φ.

Recall that a measure ν on a metric space is called **Federer** if there exists a constant $K > 0$ such that for any point x and any $r > 0$,

$$\nu(B(x, 2r)) \leq K\nu(B(x, r)).$$

Theorem 5.1. *The measure ν is Federer.*

For $\alpha \geq 0$ consider the sets Λ_α defined by

$$\Lambda_\alpha = \{\, x \in \Lambda : \; d_\nu(x) = \alpha \,\}$$

and the $f_\nu(\alpha)$-spectrum for dimensions $f_\nu(\alpha) = \dim_H \Lambda_\alpha$ (see (A.6)).

Theorem 5.2.

(1) *The pointwise dimension $d_\nu(x)$ exists for ν-almost every $x \in \Lambda$ and*

$$d_\nu(x) = h_\nu(f^1)\left(\frac{1}{\lambda_\nu^+} - \frac{1}{\lambda_\nu^-}\right) + 1,$$

λ_ν^+, λ_ν^- are positive and negative values of the Lyapunov exponent of ν (see (3.3), (3.4)).

(2) *If ν is not the measure of full dimension then the function $f_\nu(\alpha)$ is defined on an interval $[\alpha_1, \alpha_2]$ (i.e., the spectrum is* **complete***, see* [14]*); it is real analytic and strictly convex.*

(3) *If ν is not the measure of full dimension then there exists a strictly convex function $T(q)$ such that the functions $f_\nu(\alpha)$ and $T(q)$ form a Legendre transform pair (see (A.26)) and for any $q \in \mathbb{R}$ we have*

$$T(q) = -\lim_{r \to 0} \frac{\log \inf_{\mathcal{B}_r} \sum_{B \in \mathcal{B}_r} \nu(B)^q}{\log r},$$

where the infimum is taken over all finite covers $\mathcal{B}_r$ of Λ by open balls of radius r; in particular, for every $q > 1$,

$$\frac{T(q)}{1-q} = HP_q(\nu) = R_q(\nu)$$

(see (A.7), (A.8), (A.9)).

(4) *If ν is the measure of full dimension then $T(q) = (1-q)\dim_H \Lambda$ is a linear function; in addition, $f_\nu(\dim_H \Lambda) = \dim_H \Lambda$ and $f_\nu(\alpha) = 0$ for all $\alpha \neq \dim_H \Lambda$. In other words $f_\nu(\alpha)$ is a δ-function if and only if ν is the measure of full dimension.*

Remark 5.1. Consider the case when ν is not the measure of full dimension. Note that $f_\nu(\alpha) \leq \dim_H \Lambda$ for any $\alpha \in [\alpha_1, \alpha_2]$. Since $f_\nu(\alpha(q)) = T(q) + q\alpha(q)$ (see Appendix) we obtain that

$$f(\alpha(0)) = T(0) = \underline{\dim}_B \Lambda = \overline{\dim}_B \Lambda = \dim_H \Lambda$$

(see (A.1) and Theorem 4.2). Therefore, f_ν attains its maximum value $\dim_H \Lambda$ at the point $\alpha(0)$.

Differentiating the equality $f_\nu(\alpha(q)) = T(q) + q\alpha(q)$ with respect to q and using the fact that $\alpha(q) = -T'(q)$ we find that $\frac{d}{d\alpha} f_\nu(\alpha(q)) = q$ for every real q. This implies that

$$\lim_{\alpha\to\alpha_1} \frac{d}{d\alpha} f_\nu(\alpha(q)) = +\infty, \ \lim_{\alpha\to\alpha_2} \frac{d}{d\alpha} f_\nu(\alpha(q)) = -\infty, \text{ and } \frac{d}{d\alpha} f_\nu(\alpha(1)) = 1.$$

Since $T(1) = 0$ we have that $f(\alpha(1)) = \alpha(1)$. It follows that the graph of the function $f_\nu(\alpha)$ is tangent to the line with slope 1 at the point $\alpha(1)$. One can show that $\alpha(1)$ is the information dimension of ν (see [10]).

It easily follows from the above observations that $\dim_H \Lambda \in (\alpha_1, \alpha_2)$.

Another consequence of Theorem 5.2 is the following **multifractal decomposition** of a basic set Λ associated with the pointwise dimension of an equilibrium measure ν corresponding to a Hölder continuous function. Namely,

$$\Lambda = \hat{\Lambda} \cup \left(\bigcup_\alpha \Lambda_\alpha \right),$$

where Λ_α is the set of points for which the pointwise dimension takes on the value α and the **irregular part** $\hat{\Lambda}$ is the set of points with no pointwise dimension. One can show that $\hat{\Lambda} \neq \emptyset$; moreover, it is everywhere dense in Λ and $\dim_H \hat{\Lambda} = \dim_H \Lambda$ (oral communication by L. Barreira). We also have that each set Λ_α is everywhere dense in Λ.

An important manifestation of Theorem 5.2 is **multifractal decomposition** of the basic set Λ associated with the Lyapunov exponent $\lambda^+(x)$ and $\lambda^-(x)$ (see (3.3), (3.4)). We consider only positive Lyapunov exponent $\lambda^+(x)$; similar statements hold true for the negative Lyapunov exponent $\lambda^-(x)$ at points $x \in \Lambda$. We can write

$$\Lambda = \hat{L}^+ \cup \left(\bigcup_{\beta\in\mathbb{R}} L^+_\beta \right),$$

where

$$\hat{L}^+ = \{x \in \Lambda : \text{the limit in (3.3) does not exist}\}$$

is the **irregular part**, and

$$L^+_\beta = \{x \in \Lambda : \lambda^+(x) = \beta\}.$$

If ν is an ergodic measure for f^t we obtain that $\lambda^+(x) = \lambda^{(u)}_\nu$ for ν-almost every $x \in \Lambda$. Thus, the set $L^+_{\lambda^{(u)}_\nu} \neq \emptyset$.

We introduce the **dimension spectrum for (positive) Lyapunov exponents** by

$$\ell^+(\beta) = \dim_H L^+_\beta.$$

Let φ be a Hölder continuous function on Λ and ν the unique equilibrium measure for φ. Let also R be a Markov set. For any $y \in R$ we define a measure $\tilde{\nu}^{(u)}(y)$ on $W^{(u)}_{\mathrm{loc}}(y) \cap R$ as follows.

Let $\tilde{\varphi}$ be the pull back of φ to $\Lambda(A, \psi)$ by the coding map χ. The unique equilibrium measure corresponding to $\tilde{\varphi}$ is

$$\lambda_\mu = ((\mu \times m)(Y_\psi))^{-1} (\mu \times m)|_{Y_\psi},$$

where m is the Lebesgue measure on $\mathbb{R}$ and μ is the unique equilibrium measure on Σ_A corresponding to the Hölder continuous function

$$\Psi(\omega) = \int_0^{\psi(\omega)} \tilde{\varphi}(\omega, t)\, dt - P_{\Lambda(A,\psi)}(S, \tilde{\varphi})\, \psi(\omega)$$

(see Proposition 8.5).

We define the measure $\mu^{(u)}$ on Σ_A^+ such that for any cylinder $C_{i_0 \dots i_n}$ in Σ_A and its projection $C^+_{i_0 \dots i_n}$ to Σ_A^+,

$$\mu^{(u)}(C^+_{i_0 \dots i_n}) = \mu(C_{i_0 \dots i_n}). \tag{5.1}$$

Similarly, we define the measure $\mu^{(s)}$ on Σ_A^- such that for any cylinder $C_{i_{-n} \dots i_0}$ in Σ_A and its projection $C^-_{i_{-n} \dots i_0}$ to Σ_A^-,

$$\mu^{(s)}(C^-_{i_{-n} \dots i_0}) = \mu(C_{i_{-n} \dots i_0}). \tag{5.2}$$

There exist constants $K_1, K_2 > 0$ such that for every integers $m, n \geq 0$, and any $(\dots i_{-1} i_0 i_1 \dots) \in \Sigma_A$,

$$K_1 \leq \frac{\mu(C_{i_{-m} \dots i_n})}{\mu^{(s)}(C^-_{i_{-m} \dots i_0}) \times \mu^{(u)}(C^+_{i_0 \dots i_n})} \leq K_2$$

(see [10]).

Let Π be a rectangle corresponding to R (see (2.5)), and $x \in \Pi$. Denote by $\nu^{(u)}(x)$ the push forward of $\mu^{(u)}$ to $W^{(u)}_{\mathrm{loc}}(x, \Pi)$ by the coding map χ. Let $y \in R(x)$, then $W^{(u)}_{\mathrm{loc}}(y) \cap R(x)$ is naturally diffeomorphic to $W^{(u)}_{\mathrm{loc}}(x', \Pi)$ for some $x' \in \Pi$. Denote by $\tilde{\nu}^{(u)}(y)$ the push forward of $\nu^{(u)}(x')$ to $W^{(u)}_{\mathrm{loc}}(y) \cap R(x)$. Note that $\tilde{\nu}^{(u)}(y)$ is defined for every $y \in R$, and it is equivalent to the conditional measure generated by ν on $W^{(u)}_{\mathrm{loc}}(y) \cap R$ for ν-almost every $y \in R$.

There is a relation between the positive Lyapunov exponent $\lambda^+(x)$ and the pointwise dimension $d_{\nu^{(u)}_{\max}(x)}(x)$, where $\nu_{\max}$ is the measure of maximal entropy. For dynamical systems with discrete time this relation was first described by Weiss (see [15]). Notice that the measure of maximal entropy is a unique equilibrium measure corresponding to the function $\varphi = 0$.

Proposition 5.1.

$$L^+_\beta = \left\{ x \in \Lambda : \ d_{\tilde{\nu}^{(u)}_{\max}(x)}(x) = \frac{h_\Lambda(f^1)}{\beta} \right\},$$ *where* $\tilde{\nu}^{(u)}_{\max}(x)$ *is defined as above.*

Recall that we denoted by $\kappa^{(u)}$ the unique equilibrium measure corresponding to the function $-t^{(u)}a^{(u)}$, where $t^{(u)}$ is defined by (4.2). Let $\tilde{\eta}^{(u)}(x)$ be the measure on $W^{(u)}_{\rm loc}(x) \cap R(x)$ defined as above. Theorem 4.1 implies that $\tilde{\eta}^{(u)}(x)$ is the measure of full dimension. This together with Theorem 5.2 and Proposition 5.1 implies the following result.

Theorem 5.3.

(1) *If $\tilde{\nu}^{(u)}_{\max}(x)$ is not equivalent to the measure $\tilde{\eta}^{(u)}(x)$ for some $x \in \Lambda$ then the Lyapunov spectrum $\ell^+(\beta)$ is a real analytic strictly convex function on an interval $[\beta_1, \beta_2]$ containing the point*

$$\beta = h_\Lambda(f^1)/\dim_H(\Lambda \cap W^{(u)}_{loc}(x)).$$

(2) *If $\tilde{\nu}^{(u)}_{\max}(x)$ is equivalent to $\tilde{\eta}^{(u)}(x)$ for some $x \in \Lambda$ then the Lyapunov spectrum is a delta function, i.e.,*

$$\ell^+(\beta) = \begin{cases} \dim_H \Lambda, & \textit{for } \beta = h_\Lambda(f^1)/\dim_H(\Lambda \cap W^{(u)}_{loc}(x)) \\ 0, & \textit{for } \beta \neq h_\Lambda(f^1)/\dim_H(\Lambda \cap W^{(u)}_{loc}(x)). \end{cases}$$

Remark 5.2. One can show that if the measures $\tilde{\nu}^{(u)}_{\max}(x)$ and $\tilde{\eta}^{(u)}(x)$ are equivalent for some $x \in \Lambda$ then they are equivalent for all $x \in \Lambda$.

As an immediate consequence of Theorem 5.3 we obtain the following statement.

Corollary 5.1. *Assume that the measure $\tilde{\nu}^{(u)}_{\max}$ is not equivalent to the measure $\tilde{\eta}^{(u)}(x)$ for some $x \in \Lambda$ then the range of the function $\lambda^+(x)$ is an interval $[\beta_1, \beta_2]$ and for any β outside this interval the set L^+_β is empty (i.e., the spectrum is* **complete***, see* [14]*); in particular, the Lyapunov exponent attains uncountably many distinct values.*

One can also show that the set $\hat{L}^+$ is not empty and has full Hausdorff dimension (oral communication by L. Barreira).

Consider the geodesic flow on compact surface on negative curvature. Since the flow is conformal (see Section 3) the above results apply and give a complete description of the dimension spectrum for Lyapunov exponents. In particular, this spectrum is a δ-function if and only if the Liouville measure is the measure of maximal entropy and hence, the topological entropy of the flow coincide with its metric entropy (with respect to the Liouville measure). This implies that the curvature is constant.

Remark 5.3 The above results provide a complete description of the dimension spectra for pointwise dimensions and Lyapunov exponents for the time-one map of a conformal axiom A flow.

6. Moran Covers

Let $x \in \Lambda$ and Π be a rectangle containing x. We construct a special cover of the set $W^{(u)}_{\rm loc}(x,\Pi)$ which will be an "optimal" cover in computing the Hausdorff dimension and box dimensions.

Let $x \in \Gamma'(\mathfrak{T})$ and $t>0$ be a number such that $f^t x \in \mathfrak{T}$. Let also $\Pi_{f^t x}$ be the rectangle containing $f^t x$. For any point $y \in W^{(u)}_{\rm loc}(f^t x, \Pi_{f^t x})$ there exists a unique number $\tau(y)>0$ such that $f^{-\tau(y)}y \in W^{(u)}_{\rm loc}(x,\Pi)$ and the points $f^{-\tau}y$ $(0\le\tau\le\tau(y))$ and $f^{-\tau}(f^t x)$ $(0\le\tau\le t)$ visit the same rectangles in the same order. Define

$$Q(x,t) \;=\; \{\, f^{-\tau(y)}y,\;\; y \in W^{(u)}_{\rm loc}(f^t x,\Pi_{f^t x})\,\} \;\subset\; W^{(u)}_{\rm loc}(x,\Pi).$$

Lemma 6.1.

(1) *$Q(x,t)$ contains a ball in $W^{(u)}_{loc}(x,\Pi)$ of radius $\underline{r}(x,t)$ and is contained in a ball in $W^{(u)}_{loc}(x,\Pi)$ of radius $\overline{r}(x,t)$.*

(2) *There exist positive constants K_1 and K_2 independent of x and t such that for any $y \in Q(x,t)$.*

$$K_1\left(\exp\int_0^{\tau(y)} a^{(u)}(f^\tau y)\,d\tau\right)^{-1} \;\le\; \underline{r}(x,t) \;\le\; \overline{r}(x,t)$$
$$\le K_2\left(\exp\int_0^{\tau(y)} a^{(u)}(f^\tau y)\,d\tau\right)^{-1}.$$

We assume that the rectangles Π are small so that $K_2<1$.

Fix a number $r>0$. For any $y \in W^{(u)}_{\rm loc}(x,\Pi)\cap\Gamma'(\mathfrak{T})$ let $t(y)$ be the smallest number such that $f^{t(y)}y \in \mathfrak{T}$ and

$$\left(\exp\int_0^{t(y)} a^{(u)}(f^\tau y)\,d\tau\right)^{-1} \le r. \tag{6.1}$$

Among all points z such that $z \in Q(y,t(y))$ choose a point z_0 for which $t(z_0)$ is minimal. Let

$$Q(y) = Q(z_0, t(z_0)).$$

The properties of the Markov collection $\mathfrak{T}$ imply that the sets $Q(y)$ for different $y \in W^{(u)}_{\rm loc}(x,\Pi)\cap\Gamma'(\mathfrak{T})$ either coincide or overlap only along their boundaries. These sets comprise a cover of $W^{(u)}_{\rm loc}(x,\Pi)$ which we call a **Moran cover** of $W^{(u)}_{\rm loc}(x,\Pi)$ of size r.

We can also construct a Moran cover of $W^{(u)}_{\rm loc}(x,\Pi)$ using the symbolic representation of the flow (see Proposition 2.1 and Appendix). Recall that any $x\in\Lambda$ is the image under the coding map χ of a point $(\omega,t)\in\Lambda(A,\psi)$. If $y\in\mathfrak{T}$ then $y=\chi(\omega,0)$

for some $\omega \in \Sigma_A$. If a number $t > 0$ is such that $f^\tau y \notin \mathfrak{T}$ for $0 < \tau \le t$ then $f^t y = \chi(\omega, t)$.

Let $\tilde{a}^{(u)}$ and $\tilde{a}^{(s)}$ be the pull back of the functions $a^{(u)}$ and $a^{(s)}$ to $\Lambda(A, \psi)$ by the coding map χ. Let also $\mathbf{a}^{(s)}$ and $\mathbf{a}^{(u)}$ be the Hölder continuous function on Σ_A defined by

$$\mathbf{a}^{(s)}(\omega) = \exp \int_0^{\psi(\omega)} \tilde{a}^{(s)}(\omega, t)\, dt, \quad \mathbf{a}^{(u)}(\omega) = \exp \int_0^{\psi(\omega)} \tilde{a}^{(u)}(\omega, t)\, dt. \tag{6.2}$$

Choose $\hat{\omega} = (\dots i_{-1} i_0 i_1 \dots) \in \Sigma_A$ such that $x = \chi(\hat{\omega})$. We identify the set of points in Σ_A having the same past as $\hat{\omega}$ with the cylinder $C^+_{i_0} \subset \Sigma^+_A$.

Given $r > 0$ and a point $\omega \in C^+_{i_0}$ choose the number $n(\omega)$ such that

$$\prod_{k=0}^{n(\omega)-1} (\mathbf{a}^{(u)}(\sigma^k \omega))^{-1} > r, \qquad \prod_{k=0}^{n(\omega)} (\mathbf{a}^{(u)}(\sigma^k \omega))^{-1} \le r \tag{6.3}$$

(compare to (6.1)). It is easy to see that $n(\omega) \to \infty$ as $r \to 0$ uniformly in ω.

For any $\omega \in C^+_{i_0}$ consider the cylinder $C^+_{i_0 \dots i_{n(\omega)}}$. Let $C(\omega) \subset C^+_{i_0}$ be the largest cylinder set containing ω with the property that $C(\omega) = C^+_{i_0 \dots i_{n(\omega')}}$ for some $\omega' \in C(\omega)$ and $C^+_{i_0 \dots i_{n(\omega'')}} \subset C(\omega)$ for any $\omega'' \in C(\omega)$. The sets corresponding to different $\omega \in C^+_{i_0}$ either coincide or are disjoint. Thus, we obtain a cover $\mathcal{U}_r(C^+_{i_0})$ of $C^+_{i_0}$ of size r which we also call a **Moran cover**.

Similarly one can construct a Moran cover $\mathcal{U}_r(C^-_{i_0})$ of $C^-_{i_0}$ of size r.

The sets

$$Q = \chi(C), \quad C \in \mathcal{U}_r(C^+_{i_0})$$

comprise a cover of $W^{(u)}_{\mathrm{loc}}(x, \Pi)$ which is a Moran cover of $W^{(u)}_{\mathrm{loc}}(x, \Pi)$ of size r. These sets may overlap only along their boundaries.

Lemma 6.1 implies that a Moran cover has the following properties:

(6.4) Any element of the cover is contained in a ball of radius r and contains a ball of radius $K_1 r$ in $W^{(u)}_{\mathrm{loc}}(x, \Pi)$, where K_1 is a constant independent of r.

(6.5) The number of elements of the cover which intersects a ball $B(x, r) \subset W^{(u)}_{\mathrm{loc}}(x, \Pi)$ is bounded from above by a constant M independent of x and r. The number M is called the **Moran multiplicity factor**.

Let x be a point in a rectangle Π. Starting with a Moran cover of $W^{(u)}_{\mathrm{loc}}(x, \Pi)$ we will obtain a cover of the rectangle Π by the sets

$$\mathbf{Q}(y) = \bigcup_{z \in Q(y)} W^{(s)}_{\mathrm{loc}}(z, \Pi).$$

We call this cover the **extended Moran cover** corresponding to a given Moran cover. It follows from Lemma 6.1 and the construction of the sets $\mathbf{Q}(y)$ that

$$\sup_{z\in Q(y)} \left(\exp\int_0^{t(z)} a^{(u)}(f^\tau z)\,d\tau\right)^{-1} \le K_3\, r, \tag{6.6}$$

where $t(z)$ is defined by (6.1) and K_3 is a constant.

7. Proofs

Proof of Theorem 4.1. We first show that $t^{(u)} \le d := \dim_H W^{(u)}_{\mathrm{loc}}(x,\Pi)$ for any $x \in \Gamma'(\mathcal{T})$.

Fix $\varepsilon > 0$. By the definition of the Hausdorff dimension there exists a number $r > 0$ and a cover of $W^{(u)}_{\mathrm{loc}}(x,\Pi)$ by balls B_l, $l = 1,2,\dots$ of radius $r_l \le r$ such that

$$\sum_l r_l^{d+\varepsilon} \le 1.$$

For every $l > 0$ consider a Moran cover of $W^{(u)}_{\mathrm{loc}}(x,\Pi)$ of size r_l and the corresponding extended Moran cover of Π. Choose those sets from the extended cover that intersect B_l. Denote them by $\mathbf{Q}_l^{(1)},\dots,\mathbf{Q}_l^{(m(l))}$. The collection of sets $\{\mathbf{Q}_l^{(j)}\}_{j=1,\dots m(l)}^{l=1,2,\dots}$ forms a cover of Π which we denote by $\mathcal{G}$.

By (6.5), $m(l) \le M$, $\quad l = 1,2,\dots,$ where M is a Moran multiplicity factor. Using (6.6) we conclude that

$$\sum_{\mathbf{Q}_l^{(j)}\in\mathcal{G}} \sup_{z\in\mathbf{Q}_l^{(j)}} \left(\exp\int_0^{t(z)} a^{(u)}(f^\tau z)\,d\tau\right)^{-(d+\varepsilon)} \le M \sum_l (K_3\, r_l)^{d+\varepsilon} \le K_4,$$

where K_4 is a constant. The cylinders $C_l^{(j)} = \chi^{-1}\mathbf{Q}_l^{(j)}$, $\ \mathbf{Q}_l^{(j)} \in \mathcal{G}$ form a cover $\tilde{\mathcal{G}}$ of $C_{i_0} = \chi^{-1}(\Pi)$ for which

$$\sum_{C_l^{(j)}\in\tilde{\mathcal{G}}} \sup_{\omega\in C_l^{(j)}} \left(\exp\sum_{k=0}^{n(\omega)-1}\int_0^{\psi(\sigma^k\omega)} \tilde{a}^{(u)}(\sigma^k\omega,\tau)\,d\tau\right)^{-(d+\varepsilon)} \le K_4,$$

where $n(\omega)$ is defined by (6.3). Let

$$\varphi(\omega) = -(d+\varepsilon)\int_0^{\psi(\omega)} \tilde{a}^{(u)}(\omega,\tau)\,d\tau.$$

Note that the cylinders $C_l^{(j)}$ are of the form $C^+_{i_0\dots i_{n(\omega(l,j))}}$. Given a number $N > 0$ choose r so small that $n(\omega) \ge N$ for any $\omega \in \Sigma_A$. Then

$$M(C_{i_0},0,\varphi,\mathcal{U}^{(0)},N) \le \sum_{C_l^{(j)}\in\tilde{\mathcal{G}}} \sup_{\omega\in C_l^{(j)}} \left(\exp\sum_{k=0}^{n(\omega)-1}\varphi(\sigma^k\omega)\right) \le K_4,$$

where $\mathfrak{U}^{(0)}$ is the cover of Σ_A by cylinders $C_i = \{\omega \in \Sigma_A :\ \omega_0 = i\}$ (see (A.13)).

Let $\mathfrak{U}^{(k)}$ be the cover of Σ_A by cylinders $C_{i_{-k}\dots i_k}$. It follows from the definition of M that

$$M(C_{i_0}, 0, \varphi, \mathfrak{U}^{(k)}, N) \ \le\ |A|^k M(C_{i_0}, 0, \varphi, \mathfrak{U}^{(0)}, N+k) \ \le\ K_5,$$

where $|A|$ is the number of elements in the alphabet A. This implies that

$$m_c(C_{i_0}, 0, \varphi, \mathfrak{U}^{(k)}) \le K_5 \quad \text{and} \quad \tilde{P}_{C_{i_0}}(\varphi, \mathfrak{U}^{(k)}) \le 0$$

(see (A.14), (A.15)). Hence, $\tilde{P}_{C_{i_0}}(\sigma, \varphi) \le 0$ and

$$P_{\Sigma_A}(\sigma, \varphi) = \tilde{P}_{\Sigma_A}(\sigma, \varphi) = \max_{1 \le i \le n} \tilde{P}_{C_i}(\sigma, \varphi) \le 0.$$

We now estimate the topological pressure of the function $-(d+\varepsilon)\tilde{a}^{(u)}$ on $\Lambda(A, \psi)$ with respect to the suspension flow S. It is known (see [9]) that $P_{\Lambda(A,\Psi)}(S, -(d+\varepsilon)\tilde{a}^{(u)})$ is the unique real number c such that $P_{\Sigma_A}(\sigma, \varphi - c\psi) = 0$, and $P_{\Sigma_A}(\sigma, \varphi - c\psi)$ is a decreasing function over c. This implies that

$$P_{\Lambda(A,\Psi)}(S, -(d+\varepsilon)\tilde{a}^{(u)}) = c \le 0.$$

It follows that $t^{(u)} \le d+\varepsilon$. Since the inequality holds true for any $\varepsilon > 0$, we conclude that $t^{(u)} \le d$. This easily implies that $t^{(u)} \le \dim_H(U \cap \Lambda)$ for any $x \in \Lambda$ and any open set $U \subset W^{(u)}_{\text{loc}}(x)$.

We prove that $\overline{d} = \overline{\dim}_B(U) \le t^{(u)}$, where U is an open set in $W^{(u)}_{\text{loc}}(x) \cap \Lambda$. Recall that

$$\overline{d} = \limsup_{\varepsilon \to 0} \frac{\log N(U, \varepsilon)}{\log(1/\varepsilon)},$$

where $N(U, \varepsilon)$ is the maximal cardinality of an ε-separated set in U. For any $\delta > 0$ there exists a sequence $\{\varepsilon_k\}$, $\varepsilon_k \to 0$, such that $N(U, \varepsilon_k) \ge (1/\varepsilon_k)^{\overline{d}-\delta}$ for any $k > 0$.

Fix $\varepsilon > 0$. Take $\varepsilon_k < \varepsilon$ and let X_{ε_k} be an ε_k-separated set in U. For any $y \in X_{\varepsilon_k}$ let $\tau(y)$ be the number for which

$$\exp \int_0^{\tau(y)} a^{(u)}(f^\tau y) d\tau = \frac{2\varepsilon}{\varepsilon_k}.$$

We have that

$$\tau(y) \min_{\Lambda} a^{(u)} \ \le\ \log \frac{2\varepsilon}{\varepsilon_k} \ \le\ \tau(y) \max_{\Lambda} a^{(u)}.$$

It follows that

$$\tau(y) \in \left[K_6 \log \frac{1}{\varepsilon_k}, K_7 \log \frac{1}{\varepsilon_k} \right].$$

This implies that there exists a number t_k such that

$$\operatorname{card}\{\, y \in X_{\varepsilon_k} :\ \tau(y) \in [t_k - 1, t_k] \,\} \ \ge\ \frac{(1/\varepsilon_k)^{\overline{d}-\delta}}{K_8 \log(1/\varepsilon_k)}.$$

Let $E_k = \{\, y \in X_{\varepsilon_k} :\ \tau(y) \in [t_k - 1, t_k] \,\}$. If ε_k is sufficiently small we obtain

$$\operatorname{card} E_k \geq (1/\varepsilon_k)^{\overline{d}-2\delta}.$$

By construction, E_k is an (ε, t_k)-separated set in Λ. Hence,

$$\begin{aligned} Z_{t_k}(F, -(\overline{d}-2\delta)a^{(u)}, \varepsilon) &\geq \sum_{y\in E_k} \exp \int_0^{t_k} -(\overline{d}-2\delta)a^{(u)}(f^\tau y)d\tau \\ &\geq K_9 \sum_{y\in E_k} \left(\exp \int_0^{\tau(y)} a^{(u)}(f^\tau y)d\tau \right)^{-(\overline{d}-2\delta)} \\ &\geq K_9 \left(\frac{1}{\varepsilon_k}\right)^{\overline{d}-2\delta} \left(\frac{2\varepsilon}{\varepsilon_k}\right)^{-(\overline{d}-2\delta)} \geq K_{10} \end{aligned}$$

(see (A.16), (A.17), (A.18)).

Note that $t_k \to \infty$ as $k \to \infty$. Therefore,

$$P_\Lambda(F, -(\overline{d}-2\delta)a^{(u)}, \varepsilon) \geq 0 \quad \text{and} \quad P_\Lambda(F, -(\overline{d}-2\delta)a^{(u)}) \geq 0.$$

It follows that $\overline{d} - 2\delta \leq t^{(u)}$. Since the inequality holds true for any $\delta > 0$, we conclude that $\overline{d} \leq t^{(u)}$ and complete the proof of the first statement.

Since $\kappa^{(u)}$ is the unique equilibrium measure corresponding to the Hölder continuous function $-t^{(u)}a^{(u)}(x)$, we have

$$0 = P_\Lambda(-t^{(u)}a^{(u)}) = h_{\kappa^{(u)}}(f^1) - t^{(u)} \int_\Lambda a^{(u)}(y)\, d\kappa^{(u)}(y)$$

(see (4.1), (4.2), (A.21)), and the second statement follows.

We will prove the last three statements of the theorem.

Consider the function $-t^{(u)}\tilde{a}^{(u)}$ which is the pull back of the function $-t^{(u)}a^{(u)}(x)$ to $\Lambda(A,\psi)$ by the coding map χ. The unique equilibrium measure corresponding to $-t^{(u)}\tilde{a}^{(u)}$ is equal to

$$\lambda_\vartheta = ((\vartheta \times m)(Y_\psi))^{-1}(\vartheta \times m)|_{Y_\psi},$$

where ϑ is the unique equilibrium measure corresponding to the Hölder continuous function

$$-t^{(u)} \int_0^{\psi(\omega)} \tilde{a}^{(u)}(\omega, t)\, dt = -t^{(u)} \log \mathbf{a}^{(u)}(\omega)$$

on Σ_A and m is the Lebesgue measure on $\mathbb{R}$ (see (A.25), (A.26), Proposition 8.5). Since $P_{\Lambda(A,\psi)}(S, -t^{(u)}\tilde{a}^{(u)}) = 0$, we obtain that

$$P_{\Sigma_A}(\sigma, -t^{(u)} \log \mathbf{a}^{(u)}) = 0.$$

Therefore, there exist constants $K_{11}, K_{12} > 0$ such that for any $\omega \in \Sigma_A$ and any $n > 0$

$$K_{11} \le \frac{\vartheta\{\omega' :\ \omega'_i = \omega_i,\ i = 0, \dots, n\}}{\prod\limits_{k=0}^{n} \left(\mathbf{a}^{(u)}(\sigma^k \omega)\right)^{-t^{(u)}}} \le K_{12} \tag{7.1}$$

(see Proposition 8.4).

Let Π be a rectangle, and $x \in \Pi$. Let also C_{i_0} be the cylinder such that $C_{i_0} = \chi^{-1}(\Pi)$. We introduce the measure $\vartheta^{(u)}$ on Σ_A^+ such that for any cylinder $C_{i_0 \dots i_n} \subset \Sigma_A$ and its projection $C^+_{i_0 \dots i_n}$ to Σ_A^+

$$\vartheta^{(u)}(C^+_{i_0 \dots i_n}) = \vartheta(C_{i_0 \dots i_n}).$$

Let $\xi^{(u)}(x)$ is the push forward of $\vartheta^{(u)}$ to $W^{(u)}_{\text{loc}}(x, \Pi)$ by the coding map. Then $\xi^{(u)}(x)$ is equivalent to the conditional measure on $W^{(u)}_{\text{loc}}(x, \Pi)$ generated by the measure $\kappa^{(u)}$.

Let $B(y, r)$ be a ball in $W^{(u)}_{\text{loc}}(x, \Pi)$ of radius r. Consider a Moran cover of $W^{(u)}_{\text{loc}}(x, \Pi)$ of size r. Let $Q_1, \dots, Q_m$ be the elements of this cover which intersect the ball $B(y, r)$. Recall that $Q_j = C(\omega^{(j)})$ for some $\omega^{(j)} \in \Sigma_A^+$ (see Section 6). We have

$$\begin{aligned} \xi^{(u)}(B(y,r)) \ &\le \sum_{j=1}^{m} \xi^{(u)}(Q_j) \ = \sum_{j=1}^{m} \vartheta^{(u)}(C(\omega^{(j)})) \\ &\le\ K_{12} \sum_{j=1}^{m} \left(\prod_{k=0}^{n(\omega^{(j)})} \mathbf{a}^{(u)}(\omega^{(j)}) \right)^{-t^{(u)}} \ \le\ K_{12}\, M\, r^{t^{(u)}}, \end{aligned} \tag{7.2}$$

where M is the Moran multiplicity factor, that does not depend on r (see (6.3), (6.5), (7.1)).

Let $\omega = (\dots i_{-1} i_0 i_1 \dots) \in \Sigma_A$ be such that $y = \chi(\omega)$. Consider the cylinder $C^+_{i_0 \dots i_{n(\omega)}}$, where $n(\omega)$ is defined by (6.3). Then $\chi(C^+_{i_0 \dots i_{n(\omega)}})$ is contained in $B(y, r)$. Thus, by (7.1),

$$\begin{aligned} \xi^{(u)}(B(y,r)) \ &\ge \vartheta^{(u)}(C^+_{i_0 \dots i_{n(\omega)}}) \\ &\ge K_{11} \left(\prod_{k=0}^{n(\omega)} \mathbf{a}^{(u)}(\omega) \right)^{-t^{(u)}} \ \ge\ K_{13}\, r^{t^{(u)}}. \end{aligned} \tag{7.3}$$

It follows from (7.2) and (7.3) that $d_{\xi^{(u)}(x)}(y) = t^{(u)}$ for all $y \in W^{(u)}_{\text{loc}}(x, \Pi)$. This together with Proposition 8.2 implies that $\dim_H \xi^{(u)}(x) = t^{(u)}$.

Let $\mathcal{G}$ be a finite or countable cover of an open set $U \subset W^{(u)}_{\text{loc}}(x, \Pi)$ by open sets V with $\text{diam} V \le \varepsilon$. For any $V \in \mathcal{G}$ there exists a ball B such that $V \subset B$ and

$\mathrm{diam}B \le 2\,\mathrm{diam}V$. Such balls comprise a cover $\mathcal{B}$ of U. By (7.2),

$$\sum_{V\in\mathcal{G}}(\mathrm{diam}V)^{t^{(u)}} \ge \sum_{B\in\mathcal{B}}\left(\frac{\mathrm{diam}B}{2}\right)^{t^{(u)}} \ge \frac{1}{K_{12}M}\sum_{B\in\mathcal{B}}\xi^{(u)}(B) \ge K_{14}\,\xi^{(u)}(U),$$

and hence, $m_H(U,t^{(u)}) \ge K_{14}\,\xi^{(u)}(U)$ (see (A.2)).

Given $\delta > 0$ there exists $\varepsilon > 0$ such that for any cover $\mathcal{G}$ of U with $\mathrm{diam}\mathcal{G} \le \varepsilon$,

$$m_H(U,t^{(u)}) \le \sum_{V\in\mathcal{G}}(\mathrm{diam}V)^{t^{(u)}} + \delta.$$

Let $\mathcal{B}$ be a finite or countable cover of U by balls of diameter at most ε such that

$$\sum_{B\in\mathcal{B}}\xi^{(u)}(B) \le \xi^{(u)}(U) + \delta.$$

Using (7.3) we conclude that

$$\begin{aligned} m_H(U,t^{(u)}) &\le \sum_{B\in\mathcal{B}}(\mathrm{diam}B)^{t^{(u)}} + \delta \le \frac{1}{K_{13}}\sum_{B\in\mathcal{B}}\xi^{(u)}(B) + \delta \\ &\le \frac{\xi^{(u)}(U)}{K_{13}} + \left(\frac{1}{K_{13}} + 1\right)\delta. \end{aligned}$$

Since δ can be chosen arbitrarily, it follows that $m_H(U,t^{(u)}) \le \frac{1}{K_{13}}\xi^{(u)}(U)$.

Note that $W^{(u)}_{\mathrm{loc}}(x,\Pi)$ is diffeomorphic to $W^{(u)}_{\mathrm{loc}}(x) \cap R(x)$, and the push forward of $\xi^{(u)}(x)$ to $W^{(u)}_{\mathrm{loc}}(x) \cap R(x)$ is equivalent to $\eta^{(u)}(x)$. Statements 3, 4 and 5 of the theorem follow. □

Proof of Theorem 4.2. The following statement is a corollary of results by Hasselblatt [6].

Lemma 7.1. *Let F be a conformal axiom A flow on a basic set Λ. Then the weak unstable distribution $E^{(u)}\oplus X$ and the weak stable distribution $E^{(s)}\oplus X$ are Lipschitz.*

Recall that any rectangle Π lies in a small disk of co-dimension one which is transversal to the flow. The lemma implies that Π has a Lipschitz continuous local product structure. Since

$$\dim_H(W^{(u)}_{\mathrm{loc}}(x,\Pi)) = \overline{\dim}_B(W^{(u)}_{\mathrm{loc}}(x,\Pi)) = t^{(u)}, \text{ and}$$
$$\dim_H(W^{(s)}_{\mathrm{loc}}(x,\Pi)) = \overline{\dim}_B(W^{(s)}_{\mathrm{loc}}(x,\Pi)) = t^{(s)}$$

for any $x \in \Pi$, the Proposition 8.1 implies that

$$\dim_H \Pi = \underline{\dim}_B\Pi = \overline{\dim}_B\Pi = t^{(u)} + t^{(s)}.$$

The theorem follows since Λ is locally diffeomorphic to the product of a rectangle and an interval.

□

Proof of Theorem 5.1. We begin with the following observation.

Let $\tilde{\varphi}$ be the pull back of φ to $\Lambda(A,\psi)$ by the coding map χ. The unique equilibrium measure corresponding to $\tilde{\varphi}$ is equal to

$$\lambda_\mu = ((\mu \times m)(Y_\psi))^{-1}(\mu \times m)|_{Y_\psi},$$

where μ is the unique equilibrium measure corresponding to the Hölder continuous function $\log \Phi$ on Σ_A such that

$$\log \Phi(\omega) = \int_0^{\psi(\omega)} \tilde{\varphi}(\omega, t)\, dt - c\psi(\omega),$$

and $c = P_{\Lambda(A,\psi)}(S, \tilde{\varphi})$. Note that $P_\Lambda(\sigma, \log \Phi) = 0$. (See (A.25), (A.26), Proposition 8.5.)

Let us introduce the functions

$$\log \Phi^{(u)}(\omega^+) = -\lim_{n\to\infty} \log \frac{\mu(C_{i_1\ldots i_n})}{\mu(C_{i_0\ldots i_n})},$$
$$\log \Phi^{(s)}(\omega^-) = -\lim_{n\to\infty} \log \frac{\mu(C_{i_{-n}\ldots i_{-1}})}{\mu(C_{i_{-n}\ldots i_0})},$$

where $\omega^+ = (i_0 i_1 \ldots i_n \ldots) \in \Sigma_A^+$ and $\omega^- = (\ldots i_{-n} \ldots i_{-1} i_0) \in \Sigma_A^-$.

One can show that the above limits exist, the functions $\log \Phi^{(u)}$ and $\log \Phi^{(s)}$ are Hölder continuous, and they are projections to Σ_A^+ and Σ_A^- respectively of functions on Σ_A which are strictly cohomologous to $\log \Phi$ (see [10]). In particular,

$$P_{\Sigma_A^+}(\log \Phi^{(u)}) = P_{\Sigma_A^-}(\log \Phi^{(s)}) = 0.$$

We introduce the measures $\mu^{(u)}$ on Σ_A^+ and $\mu^{(s)}$ on Σ_A^- as in (5.1) and (5.2). The measures $\mu^{(u)}$ and $\mu^{(s)}$ are unique equilibrium measures corresponding to the Hölder continuous function $\log \Phi^{(u)}$ and $\log \Phi^{(s)}$ respectively (see [10]).

It follows from the definition of the equilibrium measure (see (A.12)) that

$$\begin{aligned} \int_{\Sigma_A^+} \log \Phi^{(u)}(\omega^+)\, d\mu^{(u)} \;&=\; \int_{\Sigma_A^-} \log \Phi^{(s)}(\omega^-)\, d\mu^{(s)} \;=\; \int_{\Sigma_A} \log \Phi(\omega)\, d\mu \\ &=\; -h_{\mu^{(u)}}\left(\sigma|_{\Sigma_A^+}\right) \;=\; -h_{\mu^{(s)}}\left(\sigma|_{\Sigma_A^-}\right) \;=\; -h_\mu(\sigma). \end{aligned} \tag{7.4}$$

Starting with the functions $\mathbf{a}^{(s)}$ and $\mathbf{a}^{(u)}$ one can similarly define functions $\mathbf{a}^{(ss)}$ on Σ_A^- and $\mathbf{a}^{(uu)}$ on Σ_A^+ which are projections of functions strictly cohomologous to $\mathbf{a}^{(s)}$ and $\mathbf{a}^{(u)}$ respectively.

We proceed with the proof of Theorem 5.1. Consider a rectangle Π and a point $x \in \operatorname{int}\Pi$. Let $\nu^{(u)}$ be the push forward of the measure $\mu^{(u)}$ to $W^{(u)}_{\mathrm{loc}}(x, \Pi)$ by the coding map χ. Then $\nu^{(u)}$ is equivalent to the conditional measure on $W^{(u)}_{\mathrm{loc}}(x, \Pi)$ generated by ν.

We will show that the measure $\nu^{(u)}$ is Federer.

Since $P_{\Sigma_A^+}(\log \Phi^{(u)}) = 0$ we conclude that there exist constants K_1 and K_2 such that for any $\omega \in \Sigma_A^+$

$$K_1 \le \frac{\mu^{(u)}\{\omega' : \ \omega_i' = \omega_i, \ i = 0, \dots, n\}}{\prod_{k=0}^{n} \Phi^{(u)}(\sigma^k(\omega))} \le K_2 \tag{7.5}$$

(see Proposition 8.4).

Given a number $r > 0$ consider a Moran cover of $W_{\text{loc}}^{(u)}(x, \Pi)$ of size r. Fix a point $y \in W_{\text{loc}}^{(u)}(x, \Pi)$. Let Q_0 be an element of the Moran cover that contains y. Let also $Q_0, \dots Q_m$ be the elements of the Moran cover that intersect $B(y, 2r)$. Recall that $Q_j = \chi(C(\omega^{(j)}))$ for some $\omega^{(j)} \in \Sigma_A^+$ (see Section 6). By the property (6.5) of the Moran cover, we have that $m \le \tilde{M}$, where $\tilde{M}$ is a constant independent of y and r. Since $\operatorname{diam} Q_0 < r$, we obtain

$$Q_0 \subset B(y, r) \subset B(y, 2r) \subset \bigcup_{j=0}^{m} Q_j.$$

Since $\mathbf{a}^{(u)}$ is a Hölder continuous function on Σ_A, it is easy to show that there exist positive constants L_1 and L_2 such that

$$L_1 \le \frac{\prod_{k=0}^{n(\omega^{(0)})} \left(\mathbf{a}^{(u)}(\sigma^k(\omega^{(0)}))\right)^{-1}}{\prod_{k=0}^{n(\omega^{(0)})} \left(\mathbf{a}^{(u)}(\sigma^k(\omega^{(j)}))\right)^{-1}} \le L_2,$$

where $n(\omega)$ is defined by (6.3). This implies that $|n(\omega^{(0)}) - n(\omega^{(j)})| \le K_3$, where K_3 is a constant independent of j and r. So we conclude that

$$K_4 \le \frac{\prod_{k=0}^{n(\omega^{(0)})} \Phi^{(u)}(\sigma^k(\omega^{(0)}))}{\prod_{k=0}^{n(\omega^{(j)})} \Phi^{(u)}(\sigma^k(\omega^{(j)}))} \le K_5. \tag{7.6}$$

It follows from (7.5) and (7.6) that

$$\nu^{(u)}(B(y, 2r)) \ \le \ \sum_{j=1}^{m} \nu^{(u)}(Q_j) \ = \ \sum_{j=1}^{m} \mu^{(u)}(C(\omega^{(j)}))$$

$$\le \ K_2 \sum_{j=1}^{m} \prod_{k=0}^{n(\omega^{(j)})} \Phi^{(u)}(\sigma^k(\omega^{(j)})) \ \le \ K_2 \tilde{M} \frac{1}{K_4} \prod_{k=0}^{n(\omega^{(0)})} \Phi^{(u)}(\sigma^k(\omega^{(0)}))$$

$$\le \ K_2 \tilde{M} \frac{1}{K_4} \frac{1}{K_1} \mu^{(u)}(C(\omega^{(0)})) \ = \ K_6 \, \nu^{(u)}(Q_0) \ \le \ K_6 \, \nu^{(u)}(B(y, r)).$$

Let $\nu^{(s)}$ be the push forward of $\mu^{(s)}$ to $W^{(s)}_{\mathrm{loc}}(x,\Pi)$. Arguing similarly one can prove that $\nu^{(s)}$ is Federer.

Since the measure ν is locally equivalent to the product $\nu^{(u)}\times\nu^{(s)}\times m$ (where m is the Lebesgue measure), it is also Federer. □

Proof of Theorem 5.2. First we define the "symbolic" level set. Given $0<r<1$ and $\omega\in\Sigma_A$, choose $n^-=n^-(\omega,r)$ and $n^+=n^+(\omega,r)$ such that

$$\begin{aligned}&\prod_{k=1-n^-}^{0}|\mathbf{a}^{(ss)}(\sigma^k(\omega^-))|>r, \qquad \prod_{k=-n^-}^{0}|\mathbf{a}^{(ss)}(\sigma^k(\omega^-))|\le r,\\ &\prod_{k=0}^{n^+-1}|\mathbf{a}^{(uu)}(\sigma^k(\omega^+))|^{-1}>r, \qquad \prod_{k=0}^{n^+}|\mathbf{a}^{(uu)}(\sigma^k(\omega^+))|^{-1}\le r.\end{aligned} \tag{7.7}$$

Fix a number $\tilde\alpha\ge 0$ and let $\tilde J_{\tilde\alpha}$ be the set of points ω in Σ_A for which the limit

$$\lim_{r\to 0}\left(\frac{\sum\limits_{k=-n^-}^{0}\log\Phi^{(s)}(\sigma^k(\omega^-))}{\sum\limits_{k=-n^-}^{0}\log|\mathbf{a}^{(ss)}(\sigma^k(\omega^-))|}-\frac{\sum\limits_{k=0}^{n^+}\log\Phi^{(u)}(\sigma^k(\omega^+))}{\sum\limits_{k=0}^{n^+}\log|\mathbf{a}^{(uu)}(\sigma^k(\omega^+))|}\right)$$

exists and is equal to $\tilde\alpha$.

Lemma 7.2. *Let* $\tilde\Lambda_\alpha=\{\,(\omega,t)\in\Lambda(A,\psi):\ \omega\in\tilde J_{\alpha-1}\}$. *Then* $\chi(\tilde\Lambda_\alpha)=\Lambda_\alpha$.

Proof. Let $J_{\alpha-1}=\{x\in\mathcal{T}:\ \overline{d}_{\nu^{(u)}\times\nu^{(s)}}(x)=\alpha-1\}$ and $B^{(u)}(x,r)$ a ball in $W^{(u)}_{\mathrm{loc}}(x,\Pi)$ centered at $x\in J_{\alpha-1}$. Fix x and choose $\omega=(\dots i_{-1}i_0i_1\dots)\in\Sigma_A$ such that $x=\chi(\omega)$. Consider the cylinder $C^+_{i_0\dots i_{n(\omega)}}$, where $n(\omega)$ is defined by (6.3). Let $Q^{(u)}(x,r)=\chi(C^+_{i_0\dots i_{n(\omega)}})$. We have $x\in Q^{(u)}(x,r)$ and $\operatorname{diam}Q^{(u)}(x,r)<r$. Therefore, $Q^{(u)}(x,r)\subset B^{(u)}(x,r)$. Since $Q^{(u)}(x,r)$ contains a ball of radius K_1r and $\nu^{(u)}$ is Federer, we obtain

$$\begin{aligned}\mu^{(u)}(C^+_{i_0\dots i_{n(\omega)}}) \ &= \ \nu^{(u)}(Q^{(u)}(x,r)) \ \le \ \nu^{(u)}(B^{(u)}(x,r)) \ \le\\ &K_7\,\nu^{(u)}(Q^{(u)}(x,r)) \ = \ K_7\,\mu^{(u)}(C^+_{i_0\dots i_{n(\omega)}}).\end{aligned}$$

It follows from (7.5) and (7.7) that

$$\lim_{r\to 0}\left(\frac{\log\nu^{(u)}(B^{(u)}(x,r))}{\log r}\times\frac{\sum\limits_{k=0}^{n^+}\log|\mathbf{a}^{(uu)}(\sigma^k(\omega^+))|^{-1}}{\sum\limits_{k=0}^{n^+}\log\Phi^{(u)}(\sigma^k(\omega^+))}\right)=1.$$

Arguing similarly one can show that

$$\lim_{r\to 0}\left(\frac{\log \nu^{(s)}(B^{(s)}(x,r))}{\log r}\times\frac{\sum\limits_{k=-n^-}^{0}\log|\mathbf{a}^{(ss)}(\sigma^k(\omega^-))|}{\sum\limits_{k=-n^-}^{0}\log\Phi^{(s)}(\sigma^k(\omega^-))}\right)=1,$$

where $B^{(s)}(x,r)$ are balls in $W^{(s)}_{\text{loc}}(x,\Pi)$.

This implies that $J_{\alpha-1}=\chi(\tilde J_{\alpha-1})$. Since locally Λ_α is a direct product of $J_{\alpha-1}$ and an interval, $\Lambda_\alpha=\chi(\tilde\Lambda_\alpha)$. The lemma is proven. $\square$

We proceed with the proof of Theorem 5.2. Consider the one-parameter families of functions on Σ_A

$$\begin{aligned}\varphi_q^{(u)}(\omega)&=-\tilde T^{(u)}(q)\log|\mathbf{a}^{(u)}(\omega)|+q\log\Phi(\omega),\\ \varphi_q^{(s)}(\omega)&=\tilde T^{(s)}(q)\log|\mathbf{a}^{(s)}(\omega)|+q\log\Phi(\omega),\end{aligned}\tag{7.8}$$

where $\tilde T^{(u)}(q)$ and $\tilde T^{(s)}(q)$ are chosen such that

$$P_{\Sigma_A}(\varphi_q^{(u)})=0\ \text{ and }\ P_{\Sigma_A}(\varphi_q^{(s)})=0.\tag{7.9}$$

It is known that that the functions $\tilde T^{(u)}$ and $\tilde T^{(s)}$ are real analytic (see [10]).

We introduce the functions

$$\begin{aligned}\varphi_q^{(uu)}(\omega^+)&=-\tilde T^{(u)}(q)\log|\mathbf{a}^{(uu)}(\omega^+)|+q\log\Phi^{(u)}(\omega^+),\\ \varphi_q^{(ss)}(\omega^-)&=\tilde T^{(s)}(q)\log|\mathbf{a}^{(ss)}(\omega^-)|+q\log\Phi^{(s)}(\omega^-),\end{aligned}$$

which are projections to Σ_A^+ and Σ_A^- of functions strictly cohomologous to $\varphi_q^{(u)}$ and $\varphi_q^{(s)}$ respectively.

Let $\mu_q^{(u)}$ and $\mu_q^{(s)}$ be the equilibrium measures corresponding to the Hölder continuous functions $\varphi_q^{(uu)}$ on Σ_A^+ and $\varphi^{(ss)}$ on Σ_A^- respectively.

For each real q define

$$\tilde\alpha^{(u)}(q)=-\frac{\int_{\Sigma_A^+}\log\Phi^{(u)}(\omega^+)\,d\mu_q^{(u)}}{\int_{\Sigma_A^+}\log|\mathbf{a}^{(uu)}(\omega^+)|\,d\mu_q^{(u)}},\quad \tilde\alpha^{(s)}(q)=\frac{\int_{\Sigma_A^-}\log\Phi^{(s)}(\omega^-)\,d\mu_q^{(s)}}{\int_{\Sigma_A^-}\log|\mathbf{a}^{(ss)}(\omega^-)|\,d\mu_q^{(s)}}.$$

Note that $\int_{\Sigma_A^+}\log|\mathbf{a}^{(uu)}(\omega^+)|\,d\mu_q^{(u)}>0$. The variational principle implies that

$$\int_{\Sigma_A^+}\log\Phi^{(u)}(\omega^+)\,d\mu_q^{(u)}\ \le\ P_{\Sigma_A^+}(\log\Phi^{(u)})\ =\ 0$$

(see (A.11)), and hence $\tilde\alpha^{(u)}(q)>0$ for all $q\in\mathbb{R}$. Similarly, $\tilde\alpha^{(s)}(q)>0$ for all $q\in\mathbb{R}$.

It is known that $\tilde\alpha^{(u)}(q)=-(\tilde T^{(u)})'(q)$ and $\tilde\alpha^{(s)}(q)=-(\tilde T^{(s)})'(q)$ (see [10]), in particular, $(\tilde T^{(u)})'(q)<0$ and $(\tilde T^{(s)})'(q)<0$ for all $q\in\mathbb{R}$.

Lemma 7.3.

(1) *If $\nu^{(u)}$ is the measure of full dimension then*

$$\begin{aligned} &\tilde{T}^{(u)}(q) = (1-q)\dim_H W^{(u)}_{loc}(x,\Pi), \quad \textit{and} \\ &d_{\nu^{(u)}}(y) = t^{(u)} \textit{ for all } y \in W^{(u)}_{loc}(x,\Pi), \end{aligned} \tag{7.10}$$

where $t^{(u)}$ is defined by (4.2).

(2) *If $\nu^{(u)}$ is not the measure of full dimension, then $(\tilde{T}^{(u)})''(q) > 0$ for all $q \in \mathbb{R}$.*

Proof. Recall that the conditional measure on $W^{(u)}_{\text{loc}}(x,\Pi)$ generated by the measure $\kappa^{(u)}$ is the measure of full dimension, where $\kappa^{(u)}$ is the unique equilibrium measure on Λ for the function $-t^{(u)}a^{(u)}$.

1. If $\nu^{(u)}$ is the measure of full dimension, then $\mu^{(u)}$ is the equilibrium measure for the function $-t^{(u)}\log|\mathbf{a}^{(uu)}|$, and therefore the functions $\log\Phi^{(u)}$ and $-t^{(u)}\log|\mathbf{a}^{(uu)}|$ are cohomologous (see Appendix). Since

$$P_{\Sigma_A^+}\left(\log\Phi^{(u)}\right) \;=\; P_{\Sigma_A^+}\left(-t^{(u)}\log|\mathbf{a}^{(uu)}|\right) \;=\; 0,$$

the functions are strictly cohomologous. It follows that

$$0 \;=\; P_{\Sigma_A}\left(\varphi_q^{(u)}\right) \;=\; P_{\Sigma_A^+}\left(\varphi_q^{(uu)}\right) \;=\; P_{\Sigma_A^+}\left((-\tilde{T}^{(u)}(q) - qt^{(u)})\log|\mathbf{a}^{(uu)}|\right).$$

By the definition of $t^{(u)}$ (see (4.2)), $-\tilde{T}^{(u)}(q) - q\,t^{(u)} = -t^{(u)}$, and hence $\tilde{T}^{(u)}(q) = (1-q)t^{(u)} = (1-q)\dim_H W^{(u)}_{\text{loc}}(x,\Pi)$.

The third statement of Theorem 4.1 implies that if $\nu^{(u)}$ is the measure of full dimension, then $d_{\nu^{(u)}}(y) = t^{(u)}$ for all $y \in W^{(u)}_{\text{loc}}(x,\Pi)$.

2. It is known that $(\tilde{T}^{(u)})''(q) > 0$ for some q if the functions $\log\Phi^{(u)}$ and $-(\tilde{T}^{(u)})'(q)\log|\mathbf{a}^{(uu)}|$ are not cohomologous (see [10]). Assume that the functions are cohomologous for some q. Since $(\tilde{T}^{(u)})'(q) = -\tilde{\alpha}^{(u)}(q)$, it is easy to see that

$$\int_{\Sigma_A^+}\left(\log\Phi^{(u)}(\omega^+) + (\tilde{T}^{(u)})'(q)\log|\mathbf{a}^{(uu)}(\omega^+)|\right)\,d\mu_q^{(u)} = 0.$$

This implies that the functions $\log\Phi^{(u)}$ and $-(\tilde{T}^{(u)})'(q)\log|\mathbf{a}^{(uu)}|$ are strictly cohomologous, and hence

$$P_{\Sigma_A^+}\left(-(\tilde{T}^{(u)})'(q)\log|\mathbf{a}^{(uu)}|\right) \;=\; P_{\Sigma_A^+}\left(\log\Phi^{(u)}\right) \;=\; 0$$

(see Appendix). It follows that $(\tilde{T}^{(u)})'(q) = t^{(u)}$, and $\nu^{(u)}$ is the measure of full dimension. □

Similarly to Lemma 7.3, one can prove that

(1) If $\nu^{(s)}$ is the measure of full dimension then

$$\begin{aligned}\tilde{T}^{(s)}(q) &= (1-q)\dim_H W^{(s)}_{\text{loc}}(x,\Pi), \quad \text{and}\\ d_{\nu^{(s)}}(y) &= t^{(s)} \text{ for all } y \in W^{(s)}_{\text{loc}}(x,\Pi),\end{aligned} \tag{7.11}$$

where $t^{(s)}$ is defined by (4.5).

(2) If $\nu^{(s)}$ is not the measure of full dimension, then $(\tilde{T}^{(s)})''(q) > 0$ for all $q \in \mathbb{R}$.

Set $\tilde{T}(q) = \tilde{T}^{(u)}(q) + \tilde{T}^{(s)}(q)$, and $\tilde{\alpha}(q) = \tilde{\alpha}^{(s)}(q) + \tilde{\alpha}^{(u)}(q)$. We can conclude that $\tilde{\alpha}(q) = -\tilde{T}'(q)$, in particular, $\tilde{T}' < 0$, $\tilde{T}'' \geq 0$, and $\tilde{T}'' > 0$ if and only if either $\nu^{(s)}$ or $\nu^{(u)}$ is not the measure of full dimension.

Assume that ν is not the measure of full dimension and hence, $\nu^{(s)}$ or $\nu^{(u)}$ is not the measure of full dimension.

We define the measure $\mu_q = \mu_q^{(u)} \times \mu_q^{(s)}$. Since the measures $\mu_q^{(u)}$ and $\mu_q^{(s)}$ are ergodic, it follows from the Birkhoff ergodic theorem that for μ_q-a.e. $\omega \in \Sigma_A$

$$\lim_{r\to 0}\left(\frac{\sum\limits_{k=-n^-}^{0} \log \Phi^{(s)}(\sigma^k(\omega^-))}{\sum\limits_{k=-n^-}^{0} \log |\mathbf{a}^{(ss)}(\sigma^k(\omega^-))|} - \frac{\sum\limits_{k=0}^{n^+} \log \Phi^{(u)}(\sigma^k(\omega^+))}{\sum\limits_{k=0}^{n^+} \log |\mathbf{a}^{(uu)}(\sigma^k(\omega^+))|}\right) = \tilde{\alpha}(q). \tag{7.12}$$

Lemma 7.4. *For all* $\omega = (\dots i_{-1} i_0 i_1 \dots) \in \tilde{J}_{\tilde{\alpha}(q)}$

$$\lim_{r\to 0} \frac{\log \mu_q(C_{i_{-n^-}\dots i_{n^+}})}{\log r} = \tilde{T}(q) + q\tilde{\alpha}(q),$$

where $n^- = n^-(\omega, r)$ *and* $n^+ = n^+(\omega, r)$ *are defined by (7.7).*

Proof. Since $\mu_q^{(s)}$ and $\mu_q^{(u)}$ are equilibrium measures corresponding to the functions $\varphi_q^{(ss)}$ and $\varphi^{(uu)}$, Proposition 8.4 implies that the ratios

$$\frac{\mu_q^{(s)}(C_{-i_{n^-}\dots i_0})}{\prod\limits_{k=-n^-}^{0} \mathbf{a}^{(ss)}(\sigma^k(\omega^-))^{\tilde{T}^{(s)}(q)}\, \Phi^{(s)}(\sigma^k(\omega^-))^q}$$

and

$$\frac{\mu_q^{(u)}(C_{i_0\dots i_{n^+}})}{\prod\limits_{k=0}^{n^+} \mathbf{a}^{(uu)}(\sigma^k(\omega^+))^{\tilde{T}^{(u)}(q)}\, \Phi^{(u)}(\sigma^k(\omega^+))^q}$$

are bounded from below and from above by constants independent of ω and r. Hence, for all $\omega \in \tilde{J}_{\tilde{\alpha}(q)}$,

$$\begin{aligned}
&\lim_{r\to 0} \frac{\log \mu_q(C_{i_{-n^-}\dots i_{n^+}})}{\log r} \\
&= \lim_{r\to 0} \frac{\tilde{T}^{(s)}(q) \log \prod_{k=-n^-}^{0} |\mathbf{a}^{(ss)}(\sigma^k(\omega^-))| + \tilde{T}^{(u)}(q) \log \prod_{k=0}^{n^+} |\mathbf{a}^{(uu)}(\sigma^k(\omega^+))|^{-1}}{\log r} \\
&\qquad + q \lim_{r\to 0} \left(\frac{\sum_{k=-n^-}^{0} \log \Phi^{(s)}(\sigma^k(\omega^-))}{\sum_{k=-n^-}^{0} \log |\mathbf{a}^{(ss)}(\sigma^k(\omega^-))|} - \frac{\sum_{k=0}^{n^+} \log \Phi^{(u)}(\sigma^k(\omega^+))}{\sum_{k=0}^{n^+} \log |\mathbf{a}^{(uu)}(\sigma^k(\omega^+))|} \right) \\
&= \tilde{T}^{(s)}(q) + \tilde{T}^{(u)}(q) + q\tilde{\alpha}(q).
\end{aligned}$$

The lemma is proven. □

We proceed with the proof of the theorem. Consider the measure

$$\lambda_{\mu_q} = ((\mu_q \times m)(Y_\psi))^{-1} (\mu_q \times m)|_{Y_\psi}$$

on $\Lambda(A,\psi)$. Let ν_q be its push forward. It follows from (7.12) that

$$\nu_q(\Lambda_{\tilde{\alpha}(q)+1}) = 1. \tag{7.13}$$

Similarly to the proof of Lemma 7.2 one can show that

$$d_{\nu_q}(x) = \lim_{r\to 0} \frac{\log \mu_q(C_{i_{-n^-}\dots i_{n^+}})}{\log r} + 1.$$

Lemma 7.4 implies that

$$d_{\nu_q}(x) = \tilde{T}(q) + q\tilde{\alpha}(q) + 1 \quad \text{for all } x \in \Lambda_{\tilde{\alpha}(q)+1}. \tag{7.14}$$

It follows that

$$f_\nu(\tilde{\alpha}(q)+1) = \dim_H \Lambda_{\tilde{\alpha}(q)+1} = \tilde{T}(q) + q\tilde{\alpha}(q) + 1. \tag{7.15}$$

(see [10]).

Recall that $\tilde{\alpha}(q) = -\tilde{T}'(q)$, $\tilde{T}' < 0$ and $\tilde{T}'' > 0$. Let us introduce the functions

$$\alpha(q) = \tilde{\alpha}(q) + 1 \quad \text{and} \quad T(q) = \tilde{T}(q) - q + 1.$$

We have $f_\nu(\alpha(q)) = T(q) + \alpha q$, where $\alpha(q) = -T'(q)$. Therefore, the functions f_ν and T form a Legendre transform pair (see Appendix). Clearly, the function T is real analytic, $T' < 0$, and $T'' > 0$. Therefore, f_ν is also real analytic and $f_\nu'' < 0$. The function $f_\nu(\alpha)$ is defined on an interval $[\alpha_1, \alpha_2]$, where

$$\alpha_1 = -\lim_{q\to+\infty} T'(q), \quad \alpha_2 = -\lim_{q\to-\infty} T'(q).$$

Since $P_{\Sigma_A}(\Phi)=0$, we have that $T^{(s)}(1)=T^{(u)}(1)=0$, and $\varphi^{(u)}(\omega)=\varphi^{(s)}(\omega)=\log\Phi(\omega)$ (see (7.8), (7.9)). Therefore, $\mu_1^{(s)}=\mu^{(s)}$, and $\mu_1^{(u)}=\mu^{(u)}$. It follows from the definition of $\tilde\alpha$ and (7.4) that

$$\begin{aligned}\tilde\alpha(1)&=\frac{\int_{\Sigma_A}\log\Phi(\omega)\,d\mu}{\int_{\Sigma_A}\log\mathbf{a}^{(s)}(\omega)\,d\mu}-\frac{\int_{\Sigma_A}\log\Phi(\omega)\,d\mu}{\int_{\Sigma_A}\log\mathbf{a}^{(u)}(\omega)\,d\mu}\\
&=\int_{\Sigma_A}\left(\int_0^{\psi(\omega)}\tilde\varphi(\omega,t)\,dt-c\psi(\omega)\right)d\omega\\
&\quad\times\left(\frac{1}{\int_{\Sigma_A}\int_0^{\psi(\omega)}\tilde a^{(s)}dt\,d\omega}-\frac{1}{\int_{\Sigma_A}\int_0^{\psi(\omega)}\tilde a^{(u)}dt\,d\omega}\right)\\
&=\left(K_8\int_{\Lambda(A,\psi)}\varphi(\omega,t)\,dt-c\int_{\Sigma_A}\psi(\omega)\right)\\
&\quad\times\left(\frac{1}{K_8\int_{\Lambda(A,\psi)}\tilde a^{(s)}dt\,d\omega}-\frac{1}{K_8\int_{\Lambda(A,\psi)}\tilde a^{(u)}dt\,d\omega}\right)\\
&=\left(\int_\Lambda\varphi(x)\,d\nu-c\right)\times\left(\frac{1}{\int_\Lambda a^{(s)}d\nu}-\frac{1}{\int_\Lambda a^{(u)}d\nu}\right)\\
&=h_\nu(f^1)\left(\frac{1}{\lambda_\nu^+}-\frac{1}{\lambda_\nu^-}\right),\end{aligned}$$

where $K_8=(\mu\times m)(Y_\psi)$ and $c=P_{\Lambda(A,\psi)}(S,\varphi)$.

It follows from (5.3) that μ is equivalent to μ_1, and hence ν is equivalent to ν_1. By (7.13) and (7.14), $\nu(\Lambda_{\alpha(1)})=1$. Moreover, $d_\nu(x)=\alpha(1)$ for all $x\in\Lambda_{\alpha(1)}$. This implies that

$$d_\nu(x)=h_\nu(f^1)\left(\frac{1}{\lambda_\nu^+}-\frac{1}{\lambda_\nu^-}\right)+1$$

for ν-a.e. $x\in\Lambda$. This completes the proof of the first statement.

Let $\mathcal{U}_r(C_{i_0}^+)$ and $\mathcal{U}_r(C_{i_0}^-)$ be Moran covers of $C_{i_0}^+$ and $C_{i_0}^-$ of size r. Then

$$\mathcal{C}_r=\left\{\,\mathcal{U}_r(C_{i_0}^-)\times\mathcal{U}_r(C_{i_0}^+),\ \ i_0\in A\,\right\}$$

is a cover of Σ_A. It is known that

$$\tilde T(q)=-\lim_{r\to0}\frac{\log\sum_{C\in\mathcal{C}_r}(\mu(C))^q}{\log r}.$$

Let $\tilde{\mathcal{D}}_r$ be the cover of $\Lambda(A,\psi)$ which consists of the elements

$$\tilde D=C\times[kr,(k+1)r),\quad\text{where }\ C\in\mathcal{C}_r,\ \text{ and }\ 0\le k<\frac{\max_{\omega\in C}\psi(\omega)}{r}.$$

We have that

$$-\lim_{r\to 0}\frac{\log\sum_{D\in\tilde{\mathcal{D}}_r}(\lambda_\mu(D))^q}{\log r}=\tilde{T}(q)-q+1=T(q).$$

Consider the cover $\mathcal{D}_r=\chi(\tilde{\mathcal{D}}_r)$ of Λ. By the construction there exist constants K_9 and K_{10} independent of r such that any element of $\mathcal{D}_r$ contains a ball of radius $K_9 r$ and is contained in a ball of radius $K_{10}r$.

For any $D\in\mathcal{D}_r$ consider a ball of radius $K_{10}r$ which contains D. Such balls comprise a cover $\mathcal{B}_{K_{10}r}$ of Λ. Since the measure ν is Federer,

$$\sum_{D\in\mathcal{D}_r}(\nu(D))^q\ \geq\ K_{11}\sum_{B\in\mathcal{B}_{K_{10}r}}(\nu(B))^q\,,$$

where K_{11} is a constant independent of r.

Let $\mathcal{B}_{K_9 r}$ be a cover of Λ by balls of radius $K_9 r$. For each set $D\in\mathcal{D}_r$ there exists a ball $B\in\mathcal{B}_{K_9 r}$ with the center inside D. Then the ball $\hat{B}$ of radius $2K_{10}r$ with the same center contains D. Since ν is Federer,

$$\sum_{D\in\mathcal{D}_r}(\nu(D))^q\ \leq\ \sum_{\hat{B}}(\nu(\hat{B}))^q\ \leq\ K_{12}\sum_{B\in\mathcal{B}_{K_9 r}}(\nu(B))^q,$$

where K_{12} is a constant independent of r. Therefore,

$$T(q)=-\lim_{r\to 0}\frac{\log\inf_{\mathcal{G}_r}\sum_{B\in\mathcal{B}_r}\nu(B)^q}{\log r}$$

where the infimum is taken over all finite covers $\mathcal{B}_r$ of Λ by open balls of radius r. The last part of the third statement follows now directly from the definition of $HP_q(\nu)$ (see (A.8)) and the fact that $HP_q(\nu)$ and $R_q(\nu)$ are equal (see Appendix).

If ν is the measure of full dimension, then both $\nu^{(u)}$ and $\nu^{(s)}$ are the measures of full dimension. Using (7.10), (7.11) and Theorem 4.2 we conclude that

$$\begin{aligned}&T(q)=\tilde{T}^{(s)}(q)+\tilde{T}^{(u)}(q)-q+1=(1-q)\dim_H\Lambda,\ \ \text{and}\\&d_\nu(x)=t^{(s)}+t^{(u)}+1=\dim_H\Lambda\quad\text{for all }\ x\in\Lambda.\end{aligned}$$

Hence, $f_\nu(\dim_H\Lambda)=\dim_H\Lambda$ and $f_\nu(\alpha)=0$ for $\alpha\neq\dim_H\Lambda$. This completes the proof of the theorem. □

Proof of Proposition 5.1. Recall that $\nu_{\max}$ is the unique equilibrium measure on $\Lambda(A,\psi)$ corresponding to the function $\tilde{\varphi}=0$. Therefore it is equal to λ_μ, where μ is the unique equilibrium measure on Σ_A corresponding to the Hölder continuous function

$$\Psi(\omega)=-c\psi(\omega),$$

where $c=P_{\Lambda(A,\psi)}(S,0)=P_\Lambda(F,0)=P_\Lambda(f^1,0)=h_\Lambda(f^1)$ (see (A.25), (A.26), Proposition 8.5).

Since $P_{\Sigma_A}(\Psi) = 0$, Proposition 8.4 implies that for any $\omega = (\dots i_0 i_1 \dots) \in \Sigma_A$ the ratio

$$\frac{\mu(C_{i_0\dots i_n})}{\exp \sum\limits_{k=0}^{n} \Psi(\sigma^k(\omega))}$$

is bounded from above and from below by constants independent of ω and n.

Let Π is a rectangle, $x \in \Pi \cap L_\beta^+$, and $C_{i_0} = \chi^{-1}(\Pi)$. Let $\mu^{(u)}$ be the measure on Σ_A^+ defined by (5.1), and $\nu_{\max}^{(u)}$ be the push forward of $\mu^{(u)}$ to $W_{\rm loc}^{(u)}(x, \Pi)$.

Let $B^{(u)}(x, r)$ be a ball in $W_{\rm loc}^{(u)}(x, \Pi)$. Let $\omega = (\dots i_{-1} i_0 i_1 \dots) \in \Sigma_A$ be such that $x = \chi(\omega)$. Repeating arguments in the proof of Lemma 7.2 one can show that

$$\mu^{(u)}(C^+_{i_0\dots i_{n(\omega)}}) \le \nu_{\max}^{(u)}(B^{(u)}(x, r)) \le K_{13}\, \mu^{(u)}(C^+_{i_0\dots i_{n(\omega)}}),$$

where $n(\omega)$ is defined by (6.3).

$$\begin{aligned}
d_{\nu_{\max}^{(u)}}(x) = \lim_{r\to 0} \frac{\log \nu_{\max}^{(u)}(B^{(u)}(x, r))}{\log r} \;&=\; \lim_{r\to 0} \frac{\log \mu^{(u)}(C^+_{i_0\dots i_{n(\omega)}})}{\log r} \\
=\; \lim_{r\to 0} \frac{\sum\limits_{k=0}^{n(\omega)} \Psi(\sigma^k \omega)}{\log r} \;&= \lim_{r\to 0} \frac{h_\Lambda(f^1)\, t(x)}{\int_0^{t(x)} a^{(u)}(f^\tau x)\, d\tau} \\
=\; \frac{h_\Lambda(f^1)}{\lim_{t\to\infty} \frac{1}{t}\int_0^t a^{(u)}(f^\tau x)\, d\tau} \;&=\; \frac{h_\Lambda(f^1)}{\beta},
\end{aligned}$$

where $t(x)$ is defined by (6.1). This implies that $d_{\nu_{\max}^{(u)}}(x) = h_\Lambda(f^1)/\beta$ if and only if

$$\lambda^+(x) \;=\; \lim_{t\to\infty} \frac{1}{t} \int_0^t a^{(u)}(f^\tau x)\, d\tau = \beta,$$

and the proposition follows. $\square$

Proof of Theorem 5.3. We begin with the following observation. Let $x_1, x_2 \in \Lambda$. If $x_2 = f^t(x_1)$ for some $t \in \mathbb{R}$, or $x_2 \in W_{\rm loc}^{(s)}(x_1)$, then $\lambda^+(x_1) = \lambda^+(x_2)$.

For any $x \in \Lambda$ we define the function

$$\ell_+^{(u)}(x, \beta) = \dim_H \{\, y \in W_{\rm loc}^{(u)}(x) \cap R(x) :\; \lambda^+(y) = \beta \,\},$$

where $R(x)$ is a Markov set containing x. It follows from Lemma 7.1 that this function does not depend on x, i.e. for any $x_1, x_2 \in \Lambda$

$$\ell_+^{(u)}(x_1, \beta) \;=\; \ell_+^{(u)}(x_2, \beta) \;\stackrel{\rm def}{=}\; \ell_+^{(u)}(\beta).$$

Proposition 5.1 and the proof of Theorem 5.2 imply that

(1) If $\tilde\nu_{\max}^{(u)}(x)$ is not equivalent to the measure $\tilde\eta^{(u)}(x)$ then $\ell_+^{(u)}(\beta)$ is a real analytic strictly convex function on an interval $[\beta_1, \beta_2]$.

(2) If $\tilde{\nu}^{(u)}_{\max}$ is equivalent to $\tilde{\eta}^{(u)}(x)$ then $\ell^{(u)}_+(\beta)$ is a delta function, i.e.,

$$\ell^{(u)}_+(\beta) = \begin{cases} \dim_H W^{(u)}_{\mathrm{loc}}(x), & \text{for } \beta = h_\Lambda(f^1)/\dim_H(\Lambda \cap W^{(u)}_{\mathrm{loc}}(x)) \\ 0, & \text{for } \beta \neq h_\Lambda(f^1)/\dim_H(\Lambda \cap W^{(u)}_{\mathrm{loc}}(x)). \end{cases}$$

If $\tilde{\nu}^{(u)}_{\max}(x)$ is not equivalent to the measure $\tilde{\eta}^{(u)}(x)$, an argument similar to Remark 5.1 shows that $d_{\tilde{\nu}^{(u)}_{\max}(x)}(y)$ takes on the value $\dim_H(W^{(u)}_{\mathrm{loc}}(x) \cap \Lambda)$ on a set of points $y \in W^{(u)}_{\mathrm{loc}}(x)$ of positive Hausdorff dimension. Proposition 5.1 implies that $\lambda^+(y)$ takes on the value $h_\Lambda(f^1)/\dim_H(\Lambda \cap W^{(u)}_{\mathrm{loc}}(x))$ on this set, and hence

$$\beta = h_\Lambda(f^1)/\dim_H(\Lambda \cap W^{(u)}_{\mathrm{loc}}(x)) \in (\beta_1, \beta_2).$$

Let Π be a rectangle, and $x \in \Pi$. Since

$$\begin{aligned} &\dim_H \{\, z \in W^{(u)}_{\mathrm{loc}}(x, \Pi) : \ \lambda^+(z) = \beta \,\} \\ &\qquad = \dim_H \{\, y \in W^{(u)}_{\mathrm{loc}}(x) \cap R(x) : \ \lambda^+(y) = \beta \,\} = \ell^{(u)}_+(\beta), \quad \text{and} \\ &\dim_H(W^{(s)}_{\mathrm{loc}}(x, \Pi)) = \overline{\dim}_B(W^{(s)}_{\mathrm{loc}}(x, \Pi)) \\ &\qquad = \dim_H(W^{(s)}_{\mathrm{loc}}(x)) = \overline{\dim}_B(W^{(s)}_{\mathrm{loc}}(x)) = t^{(s)} \end{aligned}$$

(see Section 4), an argument similar to the proof of Theorem 4.2 shows that

$$\ell^+(\beta) = \ell^{(u)}_+(\beta) + t^{(s)} + 1,$$

and the theorem follows. □

8. Appendix

1. Facts from dimension theory [5]. Let Z be a subset of the p-dimensional Euclidean space $\mathbb{R}^p$. The **upper box dimension** of Z is defined by

$$\overline{\dim}_B Z = \limsup_{\varepsilon \to 0} \frac{\log N(Z, \varepsilon)}{\log(1/\varepsilon)}, \qquad (A.1)$$

where $N(Z, \varepsilon)$ is the maximal cardinality of an ε-separated set in Z. The **lower box dimension** of Z, $\underline{\dim}_B Z$, is defined as the corresponding lower limit. Note that one can use $\tilde{N}(Z, \varepsilon)$, the least number of balls of radius ε needed to cover Z, instead of $N(Z, \varepsilon)$ in the above definition.

Let $\alpha \geq 0$ a number. We define the **α-Hausdorff measure** of Z by

$$m_H(Z, \alpha) = \lim_{\varepsilon \to 0} \inf_{\mathcal{G}} \sum_{U \in \mathcal{G}} (\operatorname{diam} U)^\alpha \qquad (A.2)$$

where the infimum is taken over all finite or countable coverings $\mathcal{G}$ of Z by open sets with $\operatorname{diam} \mathcal{G} \leq \varepsilon$. The **Hausdorff dimension of** Z (denoted $\dim_H Z$) is defined by

$$\dim_H Z = \inf \{\, \alpha : \ m_H(Z, \alpha) = 0 \,\} = \sup \{\, \alpha : \ m_H(Z, \alpha) = \infty \,\}. \qquad (A.3)$$

It is known that $\dim_H Z \le \underline{\dim}_B Z \le \overline{\dim}_B Z$.

The following proposition allows to compute the Hausdorff dimension and box dimensions of the Cartesian product of two sets.

Proposition 8.1. [5] *Let $U \subset \mathbb{R}^p$ and $V \subset \mathbb{R}^q$ be two Borel sets.*

(1) *If* $\dim_H U = \overline{\dim}_B U$ *then* $\dim_H(U \times V) = \dim_H U + \dim_H V$*,*
(2) *If* $\dim_H U = \overline{\dim}_B U$ *and* $\dim_H V = \overline{\dim}_B V$ *then* $\overline{\dim}_B(U \times V) = \underline{\dim}_B(U \times V) = \dim_H(U \times V) = \dim_H U + \dim_H V$*.*

Let μ be a finite Borel measure on $\mathbb{R}^p$. Its **Hausdorff dimension**, $\dim_H \mu$, is defined by

$$\dim_H \mu = \inf \{ \dim_H Z : \ \mu(Z) = 1 \}. \tag{A.4}$$

Let $K \subset \mathbb{R}^p$ be a compact subset and μ a finite Borel measure on K. The measure μ is called a **a measure of full dimension** if $\dim_H Z = \dim_H \mu$.

We now introduce the **pointwise (local) dimension** of μ at a point $x \in \mathbb{R}^p$ by

$$d_\mu(x) = \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \tag{A.5}$$

where $B(x,r)$ is the ball of radius r centered at x.

If the above limit does not exist one can consider the lower and upper limits and introduce respectively **the lower and upper pointwise dimension** of μ at x which we denote by $\underline{d}(x)$ and $\overline{d}(x)$. The functions $\underline{d}(x)$ and $\overline{d}(x)$ are measurable.

The existence of the limit in (A.5) is an important problem in dimension theory of dynamical systems. Measures for which this limit exists almost everywhere are called **exact dimensional**. The following result was established by Young in [16].

Proposition 8.2. *Let μ be a finite Borel measure on $\mathbb{R}^p$. If $d_\mu(x) = d$ for μ-almost every x then* $\dim_H \mu = d$*.*

We consider the case when μ is an invariant measure for a dynamical system.

Proposition 8.3. [1] *Let f be a $C^{1+\alpha}$ diffeomorphism of a smooth compact Riemannian manifold M, and μ an f-invariant ergodic Borel probability measure. Assume that μ is hyperbolic (i.e., all the Lyapunov exponents of f are non-zero at μ-almost every point). Then μ is exact dimensional.*

2. Dimension spectra [10]. We introduce the dimension spectrum of the measure μ which describes the distribution of values of pointwise dimension. Set

$$X_\alpha = \{ x \in \mathbb{R}^p : \ d_\mu(x) = \alpha \}.$$

The **dimension spectrum for pointwise dimensions** of the measure μ or **$f_\mu(\alpha)$-spectrum (for dimensions)** is defined by

$$f_\mu(\alpha) = \dim_H X_\alpha. \tag{A.6}$$

The straightforward calculation of the $f_\mu(\alpha)$-spectrum is difficult and one can try to relate it to another characteristics (spectra) of the invariant measure μ. Among them is the **Rényi spectrum for dimensions** defined as follows: for $q \geq 0$ set

$$R_q(\mu) = \frac{1}{q-1} \lim_{r \to 0} \frac{\log \sum_{i=1}^{N} \mu(B_i)^q}{\log r}, \tag{A.7}$$

where B_i, $i = 1, \dots, N = N(r)$ are boxes of a (uniform) grid of mesh size r (which cover the support of μ) with $\mu(B_i) > 0$ (provided the limit exists).

Another dimension spectrum is **Hentschel–Procaccia spectrum for dimensions**. It is a one-parameter family of characteristics

$$HP_q(\mu) = \frac{1}{q-1} \lim_{r \to 0} \frac{\log \inf_{\mathcal{G}} \left\{ \sum_{B(x_i,r) \in \mathcal{G}} \mu(B(x_i,r))^q \right\}}{\log r}, \tag{A.8}$$

where $\mathcal{G}$ is a finite or countable cover of the support of μ by balls of radius r and $q \geq 0$, $q \neq 1$ (provided the limit exists). One can show that for $q > 1$

$$HP_q(\mu) = \frac{1}{q-1} \lim_{r \to 0} \frac{\log \int \mu(B(x,r))^{q-1} \, d\mu(x)}{\log(1/r)}. \tag{A.9}$$

Moreover, $R_q(\mu) = HP_q(\mu)$.

3. Facts from thermodynamic formalism [3], [4], [10], [9], [13]. Let X be a compact metric space, $f \colon X \to X$ a continuous map, and φ a continuous function on X (called the **potential function**). For every $\varepsilon > 0$ and $n > 0$ a set $E \subset X$ is called (ε, n)-separated if $x, y \in E$, $x \neq y$ implies that $\rho(f^k(x), f^k(y)) > \varepsilon$ for some $k \in [0, n]$. Set

$$Z_n(f, \varphi, \varepsilon) = \sup \left\{ \sum_{x \in E} \exp \sum_{k=0}^{n-1} \varphi(f^k(x)) \right\}$$

where the supremum is taken over all (ε, n)-separated sets $E \subset X$. Set further

$$P_X(f, \varphi, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log Z_n(f, \varphi, \varepsilon),$$

$$P_X(f, \varphi) = \lim_{\varepsilon \to 0} P_X(f, \varphi, \varepsilon). \tag{A.10}$$

We call $P_X(f, \varphi)$ the **topological pressure** of the function φ on X (with respect to f).

The following result is a variational characterization of the topological pressure. Let $\mathcal{M}(f)$ denote the space of all f-invariant Borel probability measures on X. Then

$$P_X(f, \varphi) = \sup_{\mu \in \mathcal{M}(f)} \left(h_\mu(f) + \int_X \varphi \, d\mu \right) \tag{A.11}$$

where $h_\mu(f)$ is the measure-theoretic entropy of μ.

Measures that realize the variational principle for topological pressure play crucial roles in ergodic theory. A measure $\mu \in \mathcal{M}(f)$ is called an **equilibrium measure** for the function φ if

$$P_X(f,\varphi) = h_\mu(f) + \int_X \varphi \, d\mu. \tag{A.12}$$

We also need the "dimensional" definition of topological pressure for the case of a symbolic dynamical system (Σ_A, σ) (see [10]):

Let $\mathcal{U}^{(k)}$ be the open cover of Σ_A by cylinders $C_{i_{-k}\dots i_k}$. (Notice that $\operatorname{diam}\mathcal{U}^{(k)} \to 0$ as $k \to \infty$.) Let Z be a subset of Σ_A, and α be a real number. Let

$$M(Z,\alpha,\varphi,\mathcal{U}^{(k)},N) = \inf_{\tilde{\mathcal{G}}} \sum_{C \in \tilde{\mathcal{G}}} \exp\left(-\alpha(m+1) + \sup_{\omega \in C} \sum_{j=0}^{m} \varphi(\sigma^j(\omega))\right), \tag{A.13}$$

where the infimum is taken over all finite or countable collections $\mathcal{G}$ of cylinders $C = C_{i_{-k}\dots i_{k+m}}$ with $m \geq N > k$ which cover Z. Define

$$m_c(Z,\alpha,\varphi,\mathcal{U}^{(k)}) = \lim_{N\to\infty} M(Z,\alpha,\varphi,\mathcal{U}^{(k)},N), \tag{A.14}$$

$$\begin{aligned} P_Z(\varphi,\mathcal{U}^{(k)}) &= \inf\{\, \alpha : \; m_c(Z,\alpha,\varphi,\mathcal{U}^{(k)}) = 0 \,\} \\ &= \sup\{\, \alpha : \; m_c(Z,\alpha,\varphi,\mathcal{U}^{(k)}) = \infty\}, \end{aligned}$$

$$\tilde{P}_Z(f,\varphi) = \lim_{k\to\infty} P_Z(\mathcal{U}^{(k)},\varphi). \tag{A.15}$$

If Z is a compact invariant subset of Σ_A then $\tilde{P}_Z(f,\varphi) = P_Z(f,\varphi)$.

We now describe the thermodynamic formalism for dynamical systems with continuous time. Let $F = \{f^t\}\colon X \to X$ be a continuous flow (i.e., a one-parameter group of continuous maps on X which depend continuously on t) and φ a continuous function on X. For every $\varepsilon > 0$ and $t > 0$ a set $E \subset X$ is called (ε,t)-separated if $x, y \in E$, $x \neq y$ implies that $\rho(f^\tau(x), f^\tau(y)) > \varepsilon$ for some $\tau \in [0,t]$. Set

$$Z_t(F,\varphi,\varepsilon) = \sup\left\{\sum_{x\in E} \exp \int_0^t \varphi(f^\tau(x))\, d\tau\right\} \tag{A.16}$$

where the supremum is taken over all (ε,t)-separated sets $E \subset X$. Define

$$P_X(F,\varphi,\varepsilon) = \limsup_{t\to\infty} \frac{1}{t} \log Z_t(F,\varphi,\varepsilon), \tag{A.17}$$

$$P_X(F,\varphi) = \lim_{\varepsilon\to 0} P_X(F,\varphi,\varepsilon). \tag{A.18}$$

We call $P_X(F,\varphi)$ the **topological pressure** of the function φ on X (with respect to the flow $F = \{f^t\}$). One can show that

$$P_X(F,\varphi) = P_X(f^1,\varphi^1) \tag{A.19}$$

where f^1 is a time-one map and $\varphi^1 = \int_0^1 \varphi(f^t(x))\,dt$. Moreover, one can express the variational principle for the topological pressure in the case of flows as follows

$$P_X(F,\varphi) = \sup_{\mu\in\mathcal{M}(F)} \left(h_\mu(f^1) + \int_X \varphi^1\,d\mu \right), \tag{A.20}$$

where $\mathcal{M}(F)$ is the set of all F-invariant Borel probability measures on X. Note that for any such measure μ $\int \varphi^1\,d\mu = \int \varphi\,d\mu$.

A measure $\mu \in \mathcal{M}(F)$ is called an **equilibrium measure** for the function φ if

$$P_X(F,\varphi) = h_\mu(f^1) + \int_X \varphi^1\,d\mu = h_\mu(f^1) + \int_X \varphi\,d\mu. \tag{A.21}$$

4. Symbolic dynamical systems [10], [3], [4], [9]. Given a $p\times p$ matrix A of $0s$ and $1s$ (called **transfer matrix**), consider the subshift of finite type (Σ_A,σ) where Σ_A is the space of two-sided infinite sequences of p symbols which are **admissible** by the matrix A (a sequence $\omega = (\omega_i)$, $i\in\mathbb{Z}$ is admissible if $a_{\omega_i,\omega_{i+1}} = 1$ for all $i\in\mathbb{Z}$) and σ is the shift map. The space Σ_A has a natural family of metrics

$$d_\beta(\omega,\omega') = \sum_{i=-\infty}^{\infty} \frac{|\omega_i - \omega'_i|}{\beta^{|i|}} \tag{A.22}$$

where $\beta > 1$. The set Σ_A is compact with respect to the topology induced by d_β and the shift map σ is a homeomorphism. If the matrix A is **transitive** (i.e., for every $0\le i,j\le p$ there exists $k>0$ such that the (i,j)-entry of the matrix A^k is strictly positive) then the shift σ is **topologically transitive** (i.e., for every open sets U and V there exists $k>0$ such that $\sigma^k(U)\cap V\ne\emptyset$). If the matrix A is **irreducible** (i.e., there exists $k>0$ such that $A^k>0$) then the shift σ is **topologically mixing** (i.e., for every open sets U and V there exists $k>0$ such that $\sigma^n(U)\cap V\ne\emptyset$ for every $n\ge k$).

Let φ be a Hölder continuous function on Σ_A. The following statement describes equilibrium measures for subshifts of finite type.

Proposition 8.4. *Assume that the transfer matrix A is irreducible. Then*

(1) *there exists a unique equilibrium measure $\mu = \mu_\varphi$ which is mixing and is positive on open sets;*
(2) *there exist constants $D_1, D_2 > 0$ such that for any $\omega = (\omega_i)$ and any $m,n\ge 0$*

$$D_1 \le \frac{\mu\{\omega' : \omega'_i = \omega_i,\ i=-m,\dots,n\}}{\exp\left(-(m+n+1)P_{\Sigma_A}(\sigma,\varphi) + \sum_{k=-m}^{n}\varphi(\sigma^k(\omega))\right)} \le D_2. \tag{A.23}$$

A measure μ on Σ_A which satisfies (A.23) is called **a Gibbs measure**.

We describe a symbolic suspension flow over a subshift of finite type (Σ_A,σ). Let ψ be positive continuous function on Σ_A and

$$Y_\psi = \{(\omega,s) : s\in[0,\psi(\omega)],\ \omega\in\Sigma_A\} \subset \Sigma_A\times\mathbb{R}.$$

If for every $\omega \in \Sigma_A$ we identify the points $(\omega, \psi(\omega))$ and $(\sigma(\omega), 0)$ we obtain a compact topological space $\Lambda(A, \psi)$.

We define the **symbolic suspension flow** $S = \{S^t\}$ on $\Lambda(A, \psi)$ by

$$S^t(\omega, s) = (\omega, s+t) \quad \text{if } s+t \in [0, \psi(\omega)] \tag{A.24}$$

taking identification into account.

There is a canonical identification between the spaces of invariant measures for symbolic suspension flows and subshifts of finite type. Namely, for any measure $\mu \in \mathcal{M}(\sigma)$ and the Lebesgue measure m on $\mathbb{R}$ the measure $\mu \times m$ has the property that the identifications $Y_\psi \to \Lambda(A, \psi)$ are held on a set of measure zero. Therefore the measure

$$\lambda_\mu = ((\mu \times m)(Y_\psi))^{-1} (\mu \times m)|_{Y_\psi} \tag{A.25}$$

is a probability measure on $\Lambda(A, \psi)$. Moreover, $\lambda_\mu \in \mathcal{M}(S)$ and the map $\mu \to \lambda_\mu$ is one-to-one.

Let $\tilde{\varphi}$ be a continuous function on $\Lambda(A, \psi)$. Set

$$\Psi_0(\omega) = \int_0^{\psi(\omega)} \tilde{\varphi}(\omega, t)\, dt, \quad \Psi_(\omega) = \Psi_0(\omega) - c\psi(\omega) \tag{A.26}$$

where $c = P_{\Lambda(A,\psi)}(S, \tilde{\varphi})$ is the topological pressure of the function $\tilde{\varphi}$ on $\Lambda(A, \psi)$ with respect to the symbolic suspension flow S. $P_{\Sigma_A}(\sigma, \Psi) = 0$, since $P_{\Lambda(A,\psi)}(S, \tilde{\varphi})$ is the unique real number c such that $P_{\Sigma_A}(\sigma, \Psi_0 - c\psi) = 0$ (see [9]).

The following statement describes equilibrium measures for symbolic suspension flows.

Proposition 8.5. *Assume that the function $\Psi(\omega)$ is Hölder continuous on Σ_A with respect to the d_β-metric for some $\beta > 1$. Then*

(1) *there exists a unique equilibrium measure $\mu_{\tilde{\varphi}}$ for the function $\tilde{\varphi}$ for the symbolic suspension flow $S = \{S^t\}$; the measure $\mu_{\tilde{\varphi}}$ is ergodic and positive on open sets;*
(2) *$\mu_{\tilde{\varphi}} = \lambda_{\mu_\Psi}$ where μ_Ψ is a unique equilibrium measure for the function Ψ and the measure λ_{μ_Ψ} is defined by (A.25).*

5. Legendre Transform. We remind the reader of the notion of a Legendre transform pair of functions. Let h be a C^2-function on an interval I such that $h''(x) > 0$ for all $x \in I$. The Legendre transform of h is the differentiable function g of a new variable p defined by

$$g(p) = \min_{x \in I}(px + h(x)). \tag{A.27}$$

One can show that:

(1) $g'' < 0$;
(2) the Legendre transform is involutive;

(3) strictly convex functions h and g form a Legendre transform pair if and only if $g(\alpha) = h(q) + q\alpha$, where $\alpha(q) = -h'(q)$ and $q = g'(\alpha)$.

6. Cohomologous Functions [13]. Let X be a compact metric space, and $f : X \to X$ a continuous map. Two functions φ_1 and φ_2 on X are called cohomologous if there exists a Hölder continuous function $g : X \to \mathbb{R}$ and a constant K such that

$$\varphi_1 - \varphi_2 = g - g \circ f + K.$$

If the above equality holds with $K = 0$ the functions are called strictly cohomologous. We recall some properties of cohomologous functions:

(1) the functions φ_1 and φ_2 are cohomologous if and only if equilibrium measures of φ_1 and φ_2; on X coincide.
(2) if φ_1 and φ_2 are strictly cohomologous then $P_X(\varphi_1) = P_X(\varphi_2)$.

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