COCYCLES WITH ONE EXPONENT OVER PARTIALLY HYPERBOLIC SYSTEMS

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Abstract. We consider Hölder continuous linear cocycles over partially hyperbolic diffeomorphisms. For fiber bunched cocycles with one Lyapunov exponent we show continuity of measurable invariant conformal structures and sub-bundles. Further, we establish a continuous version of Zimmer's Amenable Reduction Theorem. For cocycles over hyperbolic systems we also obtain polynomial growth estimates for the norm and quasiconformal distortion from the periodic data.

1. Introduction

A linear cocycle over a dynamical system $f : \mathcal{M} \to \mathcal{M}$ is an automorphism F of a vector bundle $\mathcal{E}$ over $\mathcal{M}$ that covers f. In the case of a trivial vector bundle $\mathcal{M} \times \mathbb{R}^d$, a linear cocycle can be represented by a matrix-valued function $A : \mathcal{M} \to GL(d, \mathbb{R})$ via $F(x, v) = (f(x), A(x)v)$. In smooth dynamics linear cocycles arise naturally from the derivative. They play an important role in the study of smooth systems and group actions, especially in aspects related to rigidity.

In this paper we consider Hölder continuous linear cocycles with one Lyapunov exponent over hyperbolic and partially hyperbolic diffeomorphisms. An important motivation comes from the restriction of the derivative to Hölder continuous invariant sub-bundles such as center, stable, and unstable bundles. In hyperbolic case we studied such cocycles in [KS09, KS10]. We concentrated on obtaining conformality of the cocycle from its periodic data and applying this to local and global rigidity of Anosov systems [KS09, GKS]. From a different angle, such cocycles over hyperbolic and partially hyperbolic systems were considered in [V, ASV]. In particular, it was shown that cocycles with more than one Lyapunov exponent are generic in various cases, for example in a neighborhood of a fiber bunched cocycle. These results indicated that having one exponent is an exceptional property. In this paper we show that it is true in a very strong sense by developing a structural theory for such cocycles. We expect that these results will be useful in the study of partially hyperbolic systems and in the area of rigidity of hyperbolic systems and actions.

In the base we consider a partially hyperbolic diffeomorphism f which is is volume-preserving, accessible, and center bunched. This is the same setting as in the latest

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results on ergodicity of partially hyperbolic diffeomorphisms [BW], except that we require accessibility instead of essential accessibility. We assume that the cocycle F over f is fiber bunched, i.e. non-conformality of F in the fiber is dominated by the expansion/contraction along the stable/unstable foliations of f in the base. This or similar conditions play a role in all results on noncommutative cocycles over hyperbolic or partially hyperbolic systems. If F is the restriction of the derivative of f to the center sub-bundle, fiber bunching for F corresponds to the strong center bunching for f. Thus our results apply to this setup.

For fiber bunched cocycles with one Lyapunov exponent with respect to the volume, we prove continuity of measurable F-invariant sub-bundles and conformal structures. We use this to establish a continuous version of Zimmer's Amenable Reduction Theorem. Passing to a finite cover and a power of F, if necessary, we show existence of a continuous flag of sub-bundles such that the induced cocycles on the factor bundles are conformal. For cocycles over hyperbolic systems we obtain stronger results including Hölder regularity of the invariant structures. In particular, for cocycles with one exponent at each periodic orbit we obtain the reduction for F itself and polynomial growth estimates for its quasiconformal distortion.

We formulate the results in Section 3 and give the proofs in Section 4.

2. Definitions and notations

In this parer $\mathcal{M}$ denotes a compact connected smooth manifold.

2.1. Partially hyperbolic diffeomorphisms. (See [BW] for more details.)

A diffeomorphism f of $\mathcal{M}$ is said to be *partially hyperbolic* if there exist a nontrivial Df-invariant splitting of the tangent bundle $T\mathcal{M} = E^s \oplus E^c \oplus E^u$, and a Riemannian metric on $\mathcal{M}$ for which one can choose continuous positive functions $\nu < 1$, $\hat{\nu} < 1$, γ, $\hat{\gamma}$ such that for any $x \in \mathcal{M}$ and unit vectors $\mathbf{v}^s \in E^s(x)$, $\mathbf{v}^c \in E^c(x)$, and $\mathbf{v}^u \in E^u(x)$

$$\|Df(\mathbf{v}^s)\| < \nu(x) < \gamma(x) < \|Df(\mathbf{v}^c)\| < \hat{\gamma}(x)^{-1} < \hat{\nu}(x)^{-1} < \|Df(\mathbf{v}^u)\|. \tag{2.1}$$

The sub-bundles E^s, E^u, and E^c are called, respectively, stable, unstable, and center. E^s and E^u are tangent to the stable and unstable foliations W^s and W^u respectively. An *su*-path in $\mathcal{M}$ is a concatenation of finitely many subpaths which lie entirely in a single leaf of W^s or W^u. A partially hyperbolic diffeomorphism f is called *accessible* if any two points in $\mathcal{M}$ can be connected by an *su*-path.

We say that f is *volume-preserving* if it has an invariant probability measure μ in the measure class of a volume induced by a Riemannian metric (the density of μ is not required to be smooth). It is conjectured that any essentially accessible f is ergodic with respect to such μ. This was established if f is C^2 and center bunched [BW]. The diffeomorphism f is called *center bunched* if the functions $\nu, \hat{\nu}, \gamma, \hat{\gamma}$ can be chosen to satisfy

$$\nu < \gamma\hat{\gamma} \quad \text{and} \quad \hat{\nu} < \gamma\hat{\gamma}. \tag{2.2}$$

This implies that $\|Df|_{E^c}\| \cdot \|(Df|_{E^c})^{-1}\|$, which is a measure of non-conformality of f on E^c, is dominated by the contraction on E^s and expansion on E^u.

If f is $C^{1+\delta}$, the ergodicity holds under *strong center bunching* assumption [BW]:

$$\nu^\theta < \gamma\hat{\gamma} \quad \text{and} \quad \hat{\nu}^\theta < \gamma\hat{\gamma}, \tag{2.3}$$

where $\theta \in (0,\delta)$ satisfies

$$\nu\gamma^{-1} < \kappa^\theta \quad \text{and} \quad \hat{\nu}\hat{\gamma}^{-1} < \hat{\kappa}^\theta, \tag{2.4}$$

for some functions κ and $\hat{\kappa}$ such that for all x in $\mathcal{M}$

$$\kappa(x) < \|Df(v)\| \text{ if } v \in E^s(x) \quad \text{and} \quad \|Df(v)\| < \hat{\kappa}(x)^{-1} \text{ if } v \in E^u(x).$$

It is known that the first inequality in (2.4) implies that $E^c \oplus E^s$ is θ-Hölder, the second one yields the same for $E^c \oplus E^s$, and thus (2.4) implies that E^c is θ-Hölder.

2.2. **Hölder continuous vector bundles and linear cocycles.**

We consider a finite dimensional β-Hölder, $0 < \beta \le 1$, vector bundle $P : \mathcal{E} \to \mathcal{M}$. This means that there exists an open cover $\{U_i\}_{i=1}^k$ of $\mathcal{M}$ with coordinate systems

$$\phi_i : P^{-1}(U_i) \to U_i \times \mathbb{R}^d, \quad \phi_i(v) = (P(v), \Phi_i(v))$$

such that $\phi_j \circ \phi_i^{-1}$ is a homeomorphism and its restriction to the fiber $L_x = \Phi_j \circ \Phi_i^{-1}|_{\{x\}\times\mathbb{R}^d}$ depends β-Hölder on x, i.e. there is C such that $\|L_x - L_y\| \le C\,\text{dist}(x,y)^\beta$ for all i,j and all $x,y \in U_i \cap U_j$. One can realize $\mathcal{E}$ as a β-Hölder sub-bundle of a trivial bundle by $\phi : \mathcal{E} \to \mathcal{M} \times \mathbb{R}^{kd}$ with $\phi(v) = (P(v), \rho_1\Phi_1(v) \times ... \times \rho_k\Phi_k(v))$, where $\{\rho_i\}$ is a β-Hölder partition of unity for $\{U_i\}$.

Using such an embedding we equip $\mathcal{E}$ with the induced β-Hölder Riemannian metric, i.e. a family of inner products on the fibers, and fix an identification $I_{xy} : \mathcal{E}_x \to \mathcal{E}_y$ of fibers at nearby points. We define the latter as $\Pi_y^{-1} \circ \Pi_x$, where Π_x is the orthogonal projection in $\mathbb{R}^{kd}$ from $\mathcal{E}_x$ to the subspace which is the middle point of the unique shortest geodesic between $\mathcal{E}_x$ and $\mathcal{E}_y$ in the Grassmannian of d-dimensional subspaces. The identifications $\{I_{xy}\}$ vary β-Hölder on a neighborhood of the diagonal in $\mathcal{M} \times \mathcal{M}$ and satisfy for some constant C and any unit vector $u \in \mathcal{E}_x$

$$I_{xy} = I_{yx}^{-1}, \;\; \|I_{xy}u - u\| \le C\text{dist}(x,y)^\beta, \text{ and hence } |\|I_{xy}\| - 1| \le C\text{dist}(x,y)^\beta. \tag{2.5}$$

Let f be a diffeomorphism of $\mathcal{M}$ and $P : \mathcal{E} \to \mathcal{M}$ be a finite dimensional β-Hölder vector bundle over $\mathcal{M}$. A continuous linear cocycle over f is a homeomorphism $F : \mathcal{E} \to \mathcal{E}$ such that $P \circ F = f \circ P$ and $F_x : \mathcal{E}_x \to \mathcal{E}_{fx}$ is a linear isomorphism. Such an F is called β'-Hölder, $0 < \beta' \le \beta$, if F_x depends β'-Hölder on x, more precisely, if there exist C such that for all nearby points $x, y \in \mathcal{M}$

$$\|F_x - I_{fxfy}^{-1} \circ F_y \circ I_{xy}\| \le C\text{dist}(x,y)^{\beta'}. \tag{2.6}$$

2.3. **Conformal structures.** (See [KS10] for more details.) A conformal structure on $\mathbb{R}^d$, $d \geq 2$, is a class of proportional inner products. The space $\mathcal{C}^d$ of conformal structures on $\mathbb{R}^d$ can be identified with the space of real symmetric positive definite $d \times d$ matrices with determinant 1, which is isomorphic to $SL(d,\mathbb{R})/SO(d,\mathbb{R})$. The group $GL(d,\mathbb{R})$ acts transitively on $\mathcal{C}^d$ via $X[C] = (\det X^T X)^{-1/d} \; X^T C\, X$, and $\mathcal{C}^d$ carries a $GL(d,\mathbb{R})$-invariant Riemannian metric of non-positive curvature. The distance to the identity in this metric is

$$\text{dist}(\text{Id}, C) = \sqrt{d}/2 \cdot \left((\log \lambda_1)^2 + \cdots + (\log \lambda_d)^2 \right)^{1/2}, \tag{2.7}$$

where $\lambda_1, \ldots, \lambda_d$ are the eigenvalues of C (see [T, p.327] for more details and [Ma, p.27] for the formula).

For a vector bundle $\mathcal{E} \to \mathcal{M}$ we can consider a bundle $\mathcal{C}$ over $\mathcal{M}$ whose fiber $\mathcal{C}_x$ is the space of conformal structures on $\mathcal{E}_x$. Using a background Riemannian metric on $\mathcal{E}$, the space $\mathcal{C}_x$ can be identified with the space of symmetric positive linear operators on $\mathcal{E}_x$ with determinant 1. We equip the fibers of $\mathcal{C}$ with the Riemannian metric as above. A continuous (measurable) section of $\mathcal{C}$ is called a continuous (measurable) conformal structure on $\mathcal{E}$.

An invertible linear map $A : \mathcal{E}_x \to \mathcal{E}_y$ induces an isometry from $\mathcal{C}_x$ to $\mathcal{C}_y$ via $A(C) = (\det(A^*A))^{1/d}(A^{-1})^* C (A^{-1})$, where C is a conformal structure viewed as an operator. If $F : \mathcal{E} \to \mathcal{E}$ is a linear cocycle over f, we say that a conformal structure τ on $\mathcal{E}$ is *F-invariant* if $F(\tau(x)) = \tau(f(x))$ for all $x \in \mathcal{M}$.

3. Statements of results

Standing assumptions. *Unless stated otherwise, in this paper*
$\mathcal{M}$ is a compact connected smooth manifold;
$f : \mathcal{M} \to \mathcal{M}$ is an accessible partially hyperbolic diffeomorphism that preserves a volume μ and is either C^2 and center bunched, or $C^{1+\delta}$ and strongly center bunched;
$P : \mathcal{E} \to \mathcal{M}$ is a finite dimensional β-Hölder vector bundle over $\mathcal{M}$;
$F : \mathcal{E} \to \mathcal{E}$ is a β-Hölder linear cocycle over f.

First we establish continuity of measurable invariant conformal structures for fiber bunched cocycles. A cocycle F over a partially hyperbolic diffeomorphism f is called *fiber bunched* if for some β-Hölder norm on $\mathcal{E}$

$$\|F(x)\| \cdot \|F(x)^{-1}\| \cdot \nu(x)^\beta < 1 \quad \text{and} \quad \|F(x)\| \cdot \|F(x)^{-1}\| \cdot \hat{\nu}(x)^\beta < 1 \tag{3.1}$$

for all x in $\mathcal{M}$. This condition allows to establish convergence of certain iterates of the cocycle along the stable and unstable leaves.

Theorem 3.1. *If F is fiber bunched, then any F-invariant μ-measurable conformal structure on $\mathcal{E}$ coincides μ-a.e. with a continuous conformal structure.*

We denote the iterate $F_{f^{n-1}x} \circ ... \circ F_{fx} \circ F_x$ by F_x^n. A cocycle F is called *uniformly quasiconformal* if the quasiconformal distortion

$$K_F(x,n) \stackrel{\text{def}}{=} \|F_x^n\| \cdot \|(F_x^n)^{-1}\| \tag{3.2}$$

is uniformly bounded for all $x \in \mathcal{M}$ and $n \in \mathbb{Z}$. The cocycle is said to be *conformal* with respect to some Riemannian metric on $\mathcal{E}$ if $K_F(x,n) = 1$ for all x and n.

Corollary 3.2. *If F is uniformly quasiconformal then it preserves a continuous conformal structure on $\mathcal{E}$, equivalently, F is conformal with respect to a continuous Riemannian metric on $\mathcal{E}$.*

Next we address continuity of measurable invariant sub-bundles. If a cocycle has more than one Lyapunov exponent, then the corresponding Lyapunov sub-bundles are invariant and measurable, but not continuous in general. We show that for a fiber bunched cocycle with only one Lyapunov exponent measurable invariant sub-bundles are continuous. We denote by $\lambda_+(F,\mu)$ and $\lambda_-(F,\mu)$ the largest and smallest Lyapunov exponents of F with respect to μ. We recall that for μ almost every $x \in \mathcal{M}$,

$$\lambda_+(F,\mu) = \lim_{n\to\infty} \frac{1}{n} \log \|F_x^n\| \quad \text{and} \quad \lambda_-(F,\mu) = \lim_{n\to\infty} \frac{1}{n} \log \|(F_x^n)^{-1}\|^{-1} \tag{3.3}$$

(see [BP, Section 2.3] for more details).

Theorem 3.3. *Suppose that F is fiber bunched and $\lambda_+(F,\mu) = \lambda_-(F,\mu)$. Then any μ-measurable F-invariant sub-bundle of $\mathcal{E}$ coincides μ-a.e. with a continuous one.*

Using Theorems 3.1 and 3.3 together with Zimmer's Amenable Reduction Theorem we obtain the following description of fiber bunched cocycles with one exponent.

For any finite cover $p : \tilde{\mathcal{M}} \to \mathcal{M}$ the pullback of $\mathcal{E}$ defines a β-Hölder vector bundle $\tilde{\mathcal{E}}$ over $\tilde{\mathcal{M}}$. If $\tilde{f} : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}$ is a diffeomorphism covering f, then F lifts uniquely to a β-Hölder linear cocycle $\tilde{F} : \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$ over $\tilde{f}$ that covers F. We call such a cocycle $\tilde{F}$ a finite cover of F.

Theorem 3.4 (Continuous Amenable Reduction)**.** *Suppose that F is fiber bunched and $\lambda_+(F,\mu) = \lambda_-(F,\mu)$. Then there exists a finite cover $\tilde{F} : \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$ of F and $N \in \mathbb{N}$ such that $\tilde{F}^N$ satisfies the following property. There exist a flag of continuous $\tilde{F}^N$-invariant sub-bundles*

$$\{0\} = \tilde{\mathcal{E}}^0 \subset \tilde{\mathcal{E}}^1 \subset ... \subset \tilde{\mathcal{E}}^{k-1} \subset \tilde{\mathcal{E}}^k = \tilde{\mathcal{E}} \tag{3.4}$$

and continuous conformal structures on the factor bundles $\tilde{\mathcal{E}}^i/\tilde{\mathcal{E}}^{i-1}$, $i = 1,...,k$, invariant under the factor-cocycles induced by $\tilde{F}^N$.

The proof shows that when it is necessary to pass to a cover, the resulting cocycle $\tilde{F}^N$ preserves more than one flag as in (3.4). Their union is preserved by $\tilde{F}$ and is the lift of an invariant object for F. To illustrate this, in Section 4.6 we construct a cocycle F on $\mathcal{E} = \mathbb{T}^2 \times \mathbb{R}^2$ with no invariant μ-measurable sub-bundles or conformal structures. Its lift $\tilde{F}$ to a double cover preserves two continuous line bundles, while F preserves a continuous field of pairs of lines.

In the case when $\tilde{\mathcal{E}}_1 = \tilde{\mathcal{E}}$, the cocycle F itself is conformal on $\mathcal{E}$ with respect to some continuous Riemannian metric. This can be easily seen from the proof or deduced from the theorem using Corollary 3.2.

If there are $d = \dim \mathcal{E}_x$ continuous vector fields which give bases for all $\tilde{\mathcal{E}}^i$, then the theorem implies that $\tilde{F}^N$ is continuously cohomologous to a cocycle with values in a "standard" maximal amenable subgroup of $GL(d, \mathbb{R})$, see Remark 4.8. However, triviality of $\tilde{\mathcal{E}}$ alone is insufficient for such reduction even if $\tilde{\mathcal{E}} = \mathbb{T}^2 \times \mathbb{R}^2$ since invariant sub-bundles may be non-orientable, see [S, Section 8.1] or the example in Section 4.6.

Remark 3.5. *If f is $C^{1+\delta}$ and satisfies the strong center bunching condition, then the above results apply to $F = Df|_{E^c}$. Indeed, the condition* (2.4) *implies that E^c is θ-Hölder and* (2.3) *yields fiber bunching* (3.1) *for $F = Df|_{E^c}$.*

We also show that fiber bunching can be replaced by the following assumption (existence of the invariant volume μ is still assumed).

Corollary 3.6. *Suppose that $\lambda_+(F, \eta) = \lambda_-(F, \eta)$ for every ergodic f-invariant measure η. Then the conclusions of Theorems 3.1, 3.3, and 3.4 hold.*

This corollary relies on a certain estimate for subadditive sequences of continuous functions. In Proposition 4.9 we give a definitive version of this useful result.

We obtain Hölder continuity of the invariant structures under a stronger accessibility assumption. The diffeomorphism f is said to be *locally α-Hölder accessible* if there exists a number $L = L(f)$ such that for all sufficiently close $x, y \in \mathcal{M}$ there is an su-path $\{x = x_0,\ x_1,\ \dots,\ x_L = y\}$ such that

$$\mathrm{dist}_{W^i}(x_{i-1}, x_i) \le C\, \mathrm{dist}(x, y)^\alpha \quad \text{for } i = 1, \dots, L.$$

Here the distance between x_{i-1} and x_i is measured along the corresponding stable or unstable leaf W^i.

Corollary 3.7. *If f is locally α-Hölder accessible then the invariant conformal structures and sub-bundles in Theorems 3.1, 3.3, 3.4 and Corollaries 3.2, 3.6 are $\alpha\beta$-Hölder.*

Now we consider a special case when f is a transitive $C^{1+\delta}$, $\delta > 0$, Anosov diffeomorphism. This means that there is no center sub-bundle E^c and thus the center bunching assumption is not needed. Due to the local product structure of the stable and unstable manifolds, f is locally α-Hölder accessible with $\alpha = 1$, and hence the corollary above applies. Moreover, we can take μ to be any ergodic measure with full support and local product structure. The latter means that μ is locally equivalent to the product of its conditional measures on the local stable and unstable manifolds. An invariant volume, if it exists, has these properties. Other examples include the measure of maximal entropy and equilibrium measures of Hölder continuous potentials. For the Anosov case, analogs of Theorem 3.1, Corollary 3.2, and of a weaker version of Theorem 3.3 were obtained in [KS10].

Corollary 3.8. *Let f be a transitive $C^{1+\delta}$ Anosov diffeomorphism, μ be an f-invariant ergodic measure with full support and local product structure, and $\mathcal{M}, \mathcal{E}, F$ be as in the standing assumptions. Then Theorems 3.1, 3.3, 3.4 and Corollary 3.2 hold and the resulting invariant conformal structures and sub-bundles are β-Hölder.*

Moreover, the fiber bunching assumption in Theorems 3.1, 3.3, 3.4 can be replaced by the assumption that for every f-periodic point p the invariant measure μ_p on its orbit satisfies $\lambda_+(F,\mu_p) = \lambda_-(F,\mu_p)$.

The assumption that there is only one exponent for every periodic measure implies the same for every invariant measure [K, Theorem 1.4]. In this case we obtain further results. In the next theorem we construct an invariant flag for F itself and we normalize the cocycle and metrics so that factor cocycles are isometries.

Theorem 3.9. *Let f be a transitive $C^{1+\delta}$ Anosov diffeomorphism. Suppose that for every f-periodic point p the invariant measure μ_p on its orbit satisfies $\lambda_+(F,\mu_p) = \lambda_-(F,\mu_p)$. Then there exist a flag of β-Hölder F-invariant sub-bundles*

$$\{0\} = \mathcal{E}^0 \subset \mathcal{E}^1 \subset \ldots \subset \mathcal{E}^{j-1} \subset \mathcal{E}^j = \mathcal{E} \tag{3.5}$$

and β-Hölder Riemannian metrics on the factor bundles $\mathcal{E}^i/\mathcal{E}^{i-1}$, $i = 1, \ldots, j$, so that for some positive β-Hölder function $\phi : \mathcal{M} \to \mathbb{R}$ the factor-cocycles induced by the cocycle ϕF on $\mathcal{E}^i/\mathcal{E}^{i-1}$ are isometries.

The stronger conclusion relies on the property that the functions by which the cocycle rescales the conformal metrics on the factor bundles are cohomologus to each other. The condition that $\lambda_+(F,\eta) = \lambda_-(F,\eta)$ for every f-invariant measure η is necessary for cohomology of these functions and for the conclusion of the theorem. This condition is not known to imply the cohomology for partially hyperbolic f, and thus we do not have an analog of Theorem 3.9 for such f.

We use Theorem 3.9 to obtain uniform polynomial growth estimates of the quasi-conformal distortion $K_F(x,n)$ and the norm of F.

Theorem 3.10 (Polynomial Growth)**.** *Let f be a transitive $C^{1+\delta}$ Anosov diffeomorphism. Suppose that for every f-periodic point p the invariant measure μ_p on its orbit satisfies $\lambda_+(F,\mu_p) = \lambda_-(F,\mu_p)$. Then there exists $m < \dim \mathcal{E}_x$ and C such that*

$$K_F(x,n) \le Cn^{2m} \quad \text{for all } x \in \mathcal{M} \text{ and } n \in \mathbb{Z}.$$

Moreover, if $\lambda_+(F,\mu_p) = \lambda_-(F,\mu_p) = 0$ for every μ_p, then there exists $m < \dim \mathcal{E}_x$ and C such that

$$\|F_x^n\| \le C|n|^m \quad \text{for all } x \in \mathcal{M} \text{ and } n \in \mathbb{Z}.$$

One can take $m = j-1$, which is the number of non-trivial sub-bundles in (3.5).

4. Proofs

4.1. **Stable holonomies.** Convergence of products of the type $(F_y^n)^{-1} \circ F_x^n$ has been observed for various types group-valued cocycles whose growth is slower than the expansion/contraction in the base (see e.g. [NT, PW]). It is also related to existence of strong stable/unstable manifolds for the extended system on the bundle. We follow the notations and terminology form [V, ASV] for linear cocycles, where it is more convenient to use the following notion of holonomy.

Definition 4.1. *A* stable holonomy *for a linear cocycle $F : \mathcal{E} \to \mathcal{E}$ is a continuous map $H^s : (x, y) \mapsto H^s_{xy}$, where $x \in \mathcal{M}$, $y \in W^s(x)$, such that*

(i) H^s_{xy} *is a linear map from $\mathcal{E}_x$ to $\mathcal{E}_y$;*
(ii) $H^s_{xx} = Id$ *and* $H^s_{yz} \circ H^s_{xy} = H^s_{xz}$*;*
(iii) $H^s_{xy} = (F_y^n)^{-1} \circ H^s_{f^n x\, f^n y} \circ F^n_x$ *for all $n \in \mathbb{N}$.*

Unstable holonomy are defined similarly. The following proposition establishes existence (cf. [ASV] for trivial bundles) and some additional properties of the holonomies. We use identifications I_{xy} defined in Section 2.2 . Holonomies do not depend on the choice of identification as they are unique by (c). We denote by $W^s_{loc}(x)$ a sufficiently small ball around x in the leaf $W^s(x)$.

Proposition 4.2. *Suppose that the cocycle F is fiber bunched. Then there exists $C > 0$ such that for any $x \in \mathcal{M}$ and $y \in W^s_{loc}(x)$,*

(a) $\|(F_y^n)^{-1} \circ I_{f^n x f^n y} \circ F^n_x - I_{xy}\| \le C\, dist(x, y)^\beta$ *for every $n \in \mathbb{N}$;*

(b) $H^s_{xy} = \lim_{n\to\infty} (F_y^n)^{-1} \circ I_{f^n x f^n y} \circ F^n_x$ *satisfies (i), (ii), (iii) of Definition 4.1 and*

$$(iv)\ \|H^s_{xy} - I_{xy}\| \le C\, dist(x, y)^\beta;$$

(c) *The stable holonomy satisfying (iv) is unique.*

H^s_{xy} can be extended to any $y \in W^s(x)$ using (iii). Similarly, for $y \in W^u(x)$, the unstable holonomy H^u is obtained as $H^u_{xy} = \lim_{n\to-\infty} (F_y^n)^{-1} \circ I_{f^n x f^n y} \circ F^n_x$.

Proof. **(a)** We fix $x \in \mathcal{M}$ and denote $x_i = f^i(x)$. Then for any $y \in W^s_{loc}(x)$ we have

$$\begin{aligned}(F_y^n)^{-1} \circ I_{x_n y_n} \circ F_x^n &= (F_y^{n-1})^{-1} \circ \left((F_{y_{n-1}})^{-1} \circ I_{x_n y_n} \circ F_{x_{n-1}}\right) \circ F_x^{n-1} = \\ &= (F_y^{n-1})^{-1} \circ (I_{x_{n-1} y_{n-1}} + r_{n-1}) \circ F_x^{n-1} = \\ &= (F_y^{n-1})^{-1} \circ I_{x_{n-1} y_{n-1}} \circ F_x^{n-1} + (F_y^{n-1})^{-1} \circ r_{n-1} \circ F_x^{n-1} = \cdots = \\ &= I_{xy} + \sum_{i=0}^{n-1} (F_y^i)^{-1} \circ r_i \circ F_x^i, \quad \text{where } r_i = (F_{y_i})^{-1} \circ I_{x_{i+1} y_{i+1}} \circ F_{x_i} - I_{x_i y_i}.\end{aligned} \tag{4.1}$$

Since F is fiber bunched, there is $\theta < 1$ such that $\|F(x)\| \cdot \|F(x)^{-1}\| \cdot \nu(x)^\beta < \theta$ for every x in $\mathcal{M}$. For the function ν we denote its trajectory product by

$$\nu_i(x) = \nu(x)\nu(fx) \dots \nu(f^{i-1}x) = \nu(x_0)\nu(x_1) \dots \nu(x_{i-1}), \quad i \in \mathbb{N}.$$

Then one can estimate $\mathrm{dist}(f^n x, f^n y) \le \mathrm{dist}(x,y)\nu_n(y)$, see e.g. [BW, Lemma 1.1].

Lemma 4.3. *There is C_0 such that for every $x \in \mathcal{M}$, $y \in W^s_{loc}(x)$, and $i \ge 0$, $\|(F^i_y)^{-1}\| \cdot \| F^i_x \| \le C_0\, \theta^i\, \nu_i(y)^{-\beta}$.*

Proof. Using (2.5) and (2.6) we obtain

$$\frac{\|F_{x_k}\|}{\|F_{y_k}\|} \le \frac{\|F_{x_k} - I^{-1}_{x_{k+1}y_{k+1}} \circ F_{y_k} \circ I_{x_k y_k}\|}{\|F_{y_k}\|} + \frac{\|I^{-1}_{x_{k+1}y_{k+1}} \circ F_{y_k} \circ I_{x_k y_k}\|}{\|F_{y_k}\|} \le$$
$$\le C_1(\mathrm{dist}(x_k,y_k))^\beta + \|I^{-1}_{x_{k+1}y_{k+1}}\| \cdot \|I_{x_k y_k}\| \le 1 + C_2(\mathrm{dist}(x_k,y_k))^\beta.$$

We estimate

$$\|(F^i_y)^{-1}\| \cdot \|F^i_x\| \le \|(F_y)^{-1}\| \cdot \|(F_{y_1})^{-1}\| \cdots \|(F_{y_{i-1}})^{-1}\| \cdot \|F_x\| \cdot \|F_{x_1}\| \cdots \|F_{x_{i-1}}\|$$
$$\le \prod_{k=0}^{i-1} \|F_{y_k}\|\, \|(F_{y_k})^{-1}\| \cdot \prod_{k=0}^{i-1} \frac{\|F_{x_k}\|}{\|F_{y_k}\|} < \prod_{k=0}^{i-1} \theta\, \nu(y_k)^{-\beta} \cdot \prod_{k=0}^{i-1} \left(1 + C_2\,(\mathrm{dist}(x_k,y_k))^\beta\right).$$

Since the distance between x_n and y_n decreases exponentially, the second product is uniformly bounded and we obtain $\|(F^i_y)^{-1}\| \cdot \| F^i_x \| \le C_0\, \theta^i\, \nu_i(y)^{-\beta}$. □

Since F is Hölder continuous (2.6) we have

$$\begin{aligned} \|r_i\| = \|((F_{y_i})^{-1} \circ I_{x_{i+1}y_{i+1}} \circ F_{x_i} - I_{x_i y_i}\| \;\le\; \|(F_{y_i})^{-1} \circ I_{x_{i+1}}\| \cdot \\ \cdot \|F_{x_i} - I^{-1}_{x_{i+1}y_{i+1}} \circ F_{y_i} \circ I_{x_i y_i}\| \le C_3 \mathrm{dist}(x_i,y_i)^\beta \le C_3(C_4\, \mathrm{dist}(x,y)\, \nu_i(y))^\beta \end{aligned} \tag{4.2}$$

It follows from (4.2) and Lemma 4.3 that for every $i \ge 0$,

$$\begin{aligned} \|(F^i_y)^{-1} \circ r_i \circ F^i_x\| \le \|(F^i_y)^{-1}\| \cdot \|F^i_x\| \cdot \|r_i\| \le \\ \le C_0\, \theta^i\, \nu_i(y)^{-\beta}\, C_3 C_4^\beta\, \mathrm{dist}(x,y)^\beta \nu_i(y)^\beta = C_5\, \mathrm{dist}(x,y)^\beta\, \theta^i. \end{aligned} \tag{4.3}$$

Using (4.1), (4.3) and convergence of $\sum \theta^i$ we conclude that

$$\|(F^n_y)^{-1} \circ I_{x_n y_n} \circ F^n_x - I_{xy}\| \;\le\; \sum_{i=0}^{n-1} \|(F^i_y)^{-1} \circ r_i \circ F^i_x\| \le C\, \mathrm{dist}(x,y)^\beta.$$

(b) It follows from (4.1) that

$$\|(F^{n+1}_y)^{-1} \circ I_{x_{n+1}y_{n+1}} \circ F^{n+1}_x - (F^n_y)^{-1} \circ I_{x_n y_n} \circ F^n_x\| = \|(F^n_y)^{-1} \circ r_n \circ F^n_x\|.$$

Hence $\{(F^n_y)^{-1} \circ I_{x_n y_n} \circ F^n_x\}$ is a Cauchy sequence by (4.3), and thus it has a limit $H^s_{xy} : \mathcal{E}_x \to \mathcal{E}_y$. Since the convergence is uniform on the set of pairs (x,y) where $y \in W^s_{loc}(x)$, the map H^s is continuous. Clearly, the maps H^s_{xy} are linear and satisfy $H^s_{xx} = \mathrm{Id}$. It follows from (a) that $\|H^s_{xy} - I_{xy}\| \le C \mathrm{dist}(x,y)^\beta$. We also have

$$H^s_{xy} = \lim_{k\to\infty} (F^n_y)^{-1} \circ (F^{k-n}_{f^n y})^{-1} \circ I_{f^k x f^k y} \circ F^{k-n}_{f^n x} \circ F^n_x = (F^n_y)^{-1} \circ H^s_{f^n x\, f^n y} \circ F^n_x.$$

To show $H^s_{yz} \circ H^s_{xy} = H^s_{xz}$ we use (2.5) and Lemma 4.3 to obtain as in (4.3) that

$$\|H^s_{xz} - H^s_{yz} \circ H^s_{xy}\| \le \|(F^n_z)^{-1}\| \cdot \|(I_{x_n z_n} - I_{y_n z_n} \circ I_{x_n y_n})\| \cdot \|F^n_x\| \to 0 \quad \text{as } n \to \infty.$$

(c) Suppose that H^1 and H^2 are two stable holonomies satisfying $\|H^{1,2}_{xy} - I_{xy}\| \le C\text{dist}(x,y)^\beta$. Then using Lemma 4.3 we obtain

$$\|H^1_{xy} - H^2_{xy}\| = \|(F^n_y)^{-1} \circ (H^1_{f^nx\, f^ny} - H^1_{f^nx\, f^ny}) \circ F^n_x\| \le$$
$$\le C_0\, \theta^n\, \nu_n(y)^{-\beta}\, C\text{dist}(f^nx, f^ny)^\beta \le C_6\, \theta^n \to 0 \quad \text{as } n \to \infty,$$

and hence $H^1 = H^2$. □

4.2. **Proof of Theorem 3.1.** We use the distance between conformal structures described in Section 2.3 and the identification I from Section 2.2. We note that the proof works without change under the assumption that $\mathcal{E}$ and F are continuous and that F has continuous holonomies.

Let τ be an F-invariant μ-measurable conformal structure on $\mathcal{E}$. We first show that τ is essentially invariant under the stable and unstable holonomies of F.

Proposition 4.4. *Suppose that H^s is a stable holonomy for a linear cocycle F. If τ is a measurable F-invariant conformal structure then τ is essentially H^s-invariant, i.e. there is a set $G \subset \mathcal{M}$ of full measure such that*

$$\tau(y) = H^s_{xy}(\tau(x)) \quad \textit{for all } x, y \in G \ \textit{ such that } y \in W^s_{loc}(x).$$

Proof. We denote by $F^n_x(\xi)$ the push forward of a conformal structures ξ from $\mathcal{E}(x)$ to $\mathcal{E}(f^nx)$ induced by F^n_x, and similar notations for push forwards by H^s and I. We also let $x_i = f^i(x)$. Since τ is F-invariant, and F^n_y induces an isometry, we obtain

$$\text{dist}\,(\tau(y), H^s_{xy}(\tau(x))) = \text{dist}(F^n_y(\tau(y)), F^n_y H^s_{xy}(\tau(x))) =$$
$$= \text{dist}(\tau(y_n), H^s_{x_ny_n} F^n_x(\tau(x))) = \text{dist}(\tau(y_n), H^s_{x_ny_n}(\tau(x_n))) \le$$
$$\le \text{dist}(\tau(y_n), I_{x_ny_n}(\tau(x_n))) + \text{dist}(I_{x_ny_n}(\tau(x_n)), H^s_{x_ny_n}(\tau(x_n))).$$

Since τ is μ-measurable, by Lusin's Theorem there exists a compact set $S \subset \mathcal{M}$ with $\mu(S) > 1/2$ on which τ is uniformly continuous and hence bounded. Let G be the set of points in $\mathcal{M}$ for which the frequency of visiting S equals $\mu(S) > 1/2$. By Birkhoff Ergodic Theorem, $\mu(G) = 1$.

Suppose that both x and y are in G. Then there exists a sequence $\{n_i\}$ such that $x_{n_i} \in S$ and $y_{n_i} \in S$. Since $y \in W^s_{loc}(x)$, $\text{dist}(x_{n_i}, y_{n_i}) \to 0$ and hence $\text{dist}(I_{x_{n_i}y_{n_i}}(\tau(x_{n_i})), \tau(y_{n_i})) \to 0$ by uniform continuity of τ on S. Since H^s and I are continuous and satisfy $H^s_{xx} = \text{Id} = I_{xx}$, we have $\|I^{-1}_{x_{n_i}y_{n_i}} \circ H^s_{x_{n_i}y_{n_i}} - \text{Id}\| \to 0$. Since τ is bounded on S, the lemma below yields

$$\text{dist}(I_{x_{n_i}y_{n_i}}(\tau(x_{n_i})), H^s_{x_{n_i}y_{n_i}}(\tau(x_{n_i}))) = \text{dist}(\tau(x_{n_i}),\ I^{-1}_{x_{n_i}y_{n_i}} \circ H^s_{x_{n_i}y_{n_i}}(\tau(x_{n_i}))) \to 0.$$

We conclude that $\text{dist}\,(\tau(y), H^s_{xy}(\tau(x))) = 0$ and thus τ is essentially H^s-invariant.

Lemma 4.5. [KS10, Lemma 4.5] *Let σ be a conformal structure on $\mathbb{R}^d$ and A be a linear transformation of $\mathbb{R}^d$ sufficiently close to the identity. Then*

$$\textit{dist}\,(\sigma, A(\sigma)) \le k(\sigma) \cdot \|A - \textit{Id}\|,$$

where $k(\sigma)$ is bounded on compact sets in $\mathcal{C}^d$. More precisely, if σ is given by a matrix C, then $k(\sigma) \le 3d\,\|C^{-1}\| \cdot \|C\|$ for any A with $\|A - Id\| \le (6\|C^{-1}\| \cdot \|C\|)^{-1}$.

□

Similarly, τ is essentially H^u-invariant. Since the stable and unstable holonomies of F are continuous we conclude that τ is essentially uniformly continuous along W^s and W^u. Since the base system f is center bunched and accessible this implies continuity of τ on $\mathcal{M}$ by [ASV, Theorem E] or [W, Theorem 4.2].

□

4.3. **Proof of Proposition 3.2.** We use the following proposition from [KS10]. Recall that a measurable conformal structure τ on $\mathcal{E}$ is called *bounded* if the distance between $\tau(x)$ and $\tau_0(x)$ is essentially bounded on $\mathcal{M}$ for a continuous conformal structure τ_0 on $\mathcal{E}$.

Proposition 4.6. [KS10, Proposition 2.4] *Let f be a homeomorphism of a compact manifold $\mathcal{M}$ and let $F : \mathcal{E} \to \mathcal{E}$ be a continuous linear cocycle over f. If F is uniformly quasiconformal then it preserves a bounded measurable conformal structure τ on $\mathcal{E}$.*

Under our standing assumptions, Theorem 3.1 now implies that τ is continuous. We can normalize it by a continuous function on $\mathcal{M}$ to obtain a Riemannian metric with respect to which F is conformal. □

4.4. **Proof of Theorem 3.3.** Let $\mathcal{E}'$ be a measurable F-invariant sub-bundle of $\mathcal{E}$ with $\dim \mathcal{E}'_x = d'$. We consider a fiber bundle $\mathcal{G}$ over $\mathcal{M}$ whose fiber over x is the Grassman manifold $\mathcal{G}_x$ of all d'-dimensional subspaces in $\mathcal{E}_x$. Then F induces the cocycle $\tilde{F} : \mathcal{G} \to \mathcal{G}$ over f with diffeomorphisms $\tilde{F}_x : \mathcal{G}_x \to \mathcal{G}_{fx}$ depending continuously on x in smooth topology.

The stable holonomy H^s for F induces a stable holonomy $\tilde{H}^s$ for $\tilde{F}$. Similarly, to the linear case, this is a family of diffeomorphisms $\tilde{H}^s_{xy} : \mathcal{G}_x \to \mathcal{G}_y$ that satisfies properties (ii) and (iii) Definition 4.1 and depends continuously on x and $y \in W^s_{loc}(x)$. Similarly H^u induces the unstable holonomy $\tilde{H}^u$ for $\tilde{F}$.

The sub-bundle $\mathcal{E}'$ gives rise to a μ-measurable $\tilde{F}$-invariant section $\phi : \mathcal{M} \to \mathcal{G}$. We take m to be the lift of μ to the graph Φ of ϕ, i.e. for a set $X \subset \mathcal{G}$ we define $m(X) = \mu(\pi(X \cap \Phi))$, where $\pi : \mathcal{G} \to \mathcal{M}$ is the projection. Equivalently, m can be defined by specifying that for μ-almost every x in $\mathcal{M}$ the conditional measure m_x in the fiber $\mathcal{G}_x$ is the atomic measure at $\phi(x)$. Since μ is f-invariant and Φ is $\tilde{F}$-invariant, the measure m is $\tilde{F}$-invariant.

Lemma 4.7. [KS10, Lemma 4.6] *There exists $C > 0$ such that for any $x \in \mathcal{M}$, subspaces $\xi, \eta \in \mathcal{G}_x$, and $n \in \mathbb{Z}$*

$$dist(\tilde{F}^n_x(\xi), \tilde{F}^n_x(\eta)) \le C \cdot K_F(x,n) \cdot dist(\xi, \eta) \tag{4.4}$$

The definitions of $K(x,n)$, $\lambda_+(F,\mu)$, and $\lambda_-(F,\mu)$ yield that for μ almost all x

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\log K(x,n) &= \lim_{n\to\infty}\frac{1}{n}\log(\|F_x^n\|\cdot\|(F_x^n)^{-1}\|) = \\ \lim_{n\to\infty}\frac{1}{n}\log\|F_x^n\| &- \lim_{n\to\infty}\frac{1}{n}\log\|(F_x^n)^{-1}\|^{-1} = \lambda_+(F,\mu)-\lambda_-(F,\mu)=0.\end{aligned} \tag{4.5}$$

Hence Lemma 4.7 implies that Lyapunov exponent of $\tilde{F}$ along the fiber is zero m a.e. This together with existence of the stable and unstable holonomies for $\tilde{F}$ allows us to apply [ASV, Theorem C] to the measure m and conclude that there exists a system of conditional measures $\tilde{m}_x$ on $\mathcal{G}_x$ for m which are holonomy invariant and depend continuously on $x \in \mathcal{M}$ in the weak* topology.

Since the conditional measures m_x and $\tilde{m}_x$ coincide for all x in a set $X \subset \mathcal{M}$ of full μ measure, we see that $\tilde{m}_x = m_x$ is the atomic measure at $\phi(x)$ for all $x \in X$. Since X is dense we obtain that $\tilde{m}_x$ is atomic for all $x \in \mathcal{M}$. Indeed, for any $x \in \mathcal{M}$ we can take a sequence $X \ni x_i \to x$ and assume by compactness of $\mathcal{G}$ that $\phi(x_i)$ converge to some $\xi \in \mathcal{G}_x$. This implies that $\tilde{m}_{x_i} = m_{x_i}$ converge to the atomic measure at ξ, which therefore coincides with $\tilde{m}_x$ by continuity of the family $\{\tilde{m}_x\}$. Denoting $\tilde{\phi}(x) = \operatorname{supp} \tilde{m}_x$ for $x \in \mathcal{M}$, we obtain a continuous section $\tilde{\phi}$ which coincides with ϕ on X. This shows that $\mathcal{E}'$ coincides μ-almost everywhere with a continuous subbundle which is invariant under the stable and unstable holonomies. $\square$

4.5. **Proof of Theorem 3.4.** We use the following particular case of Zimmer's Amenable Reduction Theorem:

[HKt, Corollary 1.8], [BP, Theorem 3.5.9] *Let f be an ergodic transformation of a measure space (X,μ) and let $F : X \to GL(d,\mathbb{R})$ be a measurable function. Then there exists a measurable function $C : X \to GL(d,\mathbb{R})$ such that the function $A(x) = C^{-1}(fx)F(x)C(x)$ takes values in an amenable subgroup of $GL(d,\mathbb{R})$.*

There are 2^{d-1} standard maximal amenable subgroups of $GL(d,\mathbb{R})$. They correspond to the distinct compositions of d, $d_1 + \cdots + d_k = d$, and each group consists of all block-triangular matrices of the form

$$\begin{bmatrix} A_1 & * & \dots & * \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & A_k \end{bmatrix} \tag{4.6}$$

where each diagonal block A_i is a scalar multiple of a $d_i \times d_i$ orthogonal matrix. Any amenable subgroup of $GL(d,\mathbb{R})$ has a finite index subgroup which is contained in a conjugate of one of these standard subgroups [M, Theorem 3.4]. The normalizer of the diagonal subgroup is an example of an amenable group which does not lie in any such conjugate. It is the finite extension of the diagonal subgroup that contains all permutations of the coordinate axes.

Since $\mathcal{E}$ can be trivialized on a set of full μ-measure [BP, Proposition 2.1.2], we can measurably identify $\mathcal{E}$ with $\mathcal{M} \times \mathbb{R}^d$ and view F as a function $\mathcal{M} \to GL(d, \mathbb{R})$. Thus can we apply the Amenable Reduction Theorem to F and obtain a measurable coordinate change function $C : \mathcal{M} \to GL(d, \mathbb{R})$ such that $A(x) = C^{-1}(fx)F(x)\, C(x) \in G$ for μ-a.e. $x \in \mathcal{M}$, where G is an amenable subgroup of $GL(d, \mathbb{R})$. By the above we may assume that G contains a finite index subgroup G_0 which is contained in one of the 2^{d-1} standard maximal amenable subgroups.

Case 1. First we consider the case when G itself is contained in a standard subgroup. Then the conclusion of the theorem holds for F itself rather than for a power of its lift. The sub-bundle V^i spanned by the first $d_1 + \cdots + d_i$ coordinate vectors in $\mathbb{R}^d$ is A-invariant for $i = 1, \dots, k$. Denoting $\mathcal{E}^i_x = C(x)V^i$ we obtain the corresponding flag of measurable F-invariant sub-bundles

$$\mathcal{E}^1 \subset \mathcal{E}^2 \subset \cdots \subset \mathcal{E}^k = \mathcal{E} \quad \text{with} \quad \dim \mathcal{E}^i = d_1 + \cdots + d_i.$$

By Theorem 3.3 we may assume that the sub-bundles $\mathcal{E}^i$ are continuous. Since $A_1(x)$ is a scalar multiple of a $d_1 \times d_1$ orthogonal matrix for μ-a.e. x, we conclude that the restriction of F to $\mathcal{E}^1$ is conformal with respect to the push forward by C of the standard conformal structure on V^1. This gives a measurable F-invariant conformal structure τ_1 on $\mathcal{E}^1$. Since F preserves $\mathcal{E}^1$, so does the stable holonomy H^s. Hence H^s induces a stable holonomy $H^{s,1}$ for the restriction of F to $\mathcal{E}^1$. By Proposition 4.4 the conformal structure τ_1 is essentially invariant under $H^{s,1}$. Similarly we obtain essential invariance of τ_1 under the unstable holonomy $H^{u,1}$. This yields continuity of τ_1 on $\mathcal{M}$ as in the end of the proof of Theorem 3.1.

Similarly, we can consider continuous factor-bundle $\mathcal{E}^i/\mathcal{E}^{i-1}$ over $\mathcal{M}$ with the natural induced cocycle $F^{(i)}$. Since the matrix of the map induced by A on $V^i/V^{i-1} = \mathbb{R}^{d_i}$ is A_i, it preserves the standard conformal structure on $\mathbb{R}^{d_i}$. Pushing forward by C we obtain a measurable conformal structure τ_i on $\mathcal{E}^i/\mathcal{E}^{i-1}$ invariant under $F^{(i)}$. The holonomies H^s and H^u induce continuous holonomies for $F^{(i)}$ on $\mathcal{E}^i/\mathcal{E}^{i-1}$. As above, we conclude that τ_i is essentially invariant under these holonomies and continuous.

Case 2. Now we consider the case when only a finite index subgroup G_0 of G is contained in a standard maximal amenable subgroup. We again consider the flag of subspaces V^i spanned by the first $n_i = d_1 + \cdots + d_i$ coordinate vectors in $\mathbb{R}^d$, $i = 1, \dots, k$. Let G_* be the stabilizer of this flag in G. Then G_* contains G_0 and thus G_* has a finite finite index l in G. The orbit of this flag under G consists of l distinct flags in $\mathbb{R}^d$ which we denote by

$$W^j = \{V^{j,1} \subset ... \subset V^{j,k-1} \subset V^{j,k} = \mathbb{R}^d\}, \qquad j = 1, ..., l. \tag{4.7}$$

Any $g \in G$ permutes these flags and preserves their union. First we will construct corresponding flags of continuous invariant sub-bundles. If $l \geq 2$ this requires in general to pass to a finite cover and a power of F. After that we will show existence of continuous invariant conformal structures on the factor bundles for each flag.

For each $i = 1, ..., k-1$ the subspaces $V^{j,i}$, $j = 1, ..., l$, have dimension n_i. Some of them may coincide, so we denote the number of distinct ones by l_i. We also denote their union and its image under C by

$$(4.8) \qquad U^{(i)} = V^{1,i} \cup ... \cup V^{l,i} \subset \mathbb{R}^d \quad \text{and} \quad \hat{\mathcal{U}}_x^{(i)} = C(x) U^{(i)} \subset \mathcal{E}_x \quad \text{for } \mu\text{-a.e. } x$$

Then $\hat{\mathcal{U}}^{(i)}$ depends measurably on x and is a union of l_i distinct n_i-dimensional subspaces of $\mathcal{E}_x$. Since $g(U^{(i)}) = U^{(i)}$ for any $g \in G$, we see that $\hat{\mathcal{U}}_x^{(i)}$ is invariant under F, i.e. $F_x(\hat{\mathcal{U}}_x^{(i)}) = \hat{\mathcal{U}}_{fx}^{(i)}$. First we claim that for each x in $\mathcal{M}$ there exists $\mathcal{U}_x^{(i)}$, a union of l_i distinct n_i-dimensional subspaces of $\mathcal{E}_x$, which depends continuously on x, coincides with $\hat{\mathcal{U}}_x^{(i)}$ for μ-a.e. x, and is invariant under F. This can be seen as in the proof of Theorem 3.3. Indeed, we can define the measure m on the corresponding Grassmannian bundle $\mathcal{G}$ by choosing the conditional measure $\hat{m}_x$ on $\mathcal{G}_x$ to be the atomic measure equidistributed on the l_i points corresponding to $\hat{\mathcal{U}}_x^{(i)}$. Then m is invariant under the induced cocycle on $\mathcal{G}$. Thus by the same argument we obtain that there exists a system of conditional measures m_x on $\mathcal{G}_x$ for m which are holonomy invariant and depend continuously on $x \in M$ in the weak* topology. As before, the measures m_x are atomic for all x. Moreover, the number of atoms is preserved by the stable and unstable holonomies since they are induced by linear isomorphisms. Hence accessibility implies that the number of atoms is l_i for all x in $\mathcal{M}$. Then m_x corresponds to a union $\mathcal{U}_x^{(i)}$ of l_i distinct n_i-dimensional subspaces in $\mathcal{E}_x$ which depends continuously on x and is invariant under F. We also denote $\mathcal{U}^{(i)} = \bigcup_{x \in \mathcal{M}} \mathcal{U}_x^{(i)} \subset \mathcal{E}$.

We fix a point $q \in \mathcal{M}$ and denote the subspaces in $\mathcal{U}_q^{(i)}$ by $\mathcal{U}_q^{1,i}, ..., \mathcal{U}_q^{l_i,i}$. Locally they can be uniquely extended to continuous sub-bundles $\mathcal{U}^{1,i}, ..., \mathcal{U}^{l_i,i}$ so that the union of $\mathcal{U}_x^{1,i}, ..., \mathcal{U}_x^{l_i,i}$ is $\mathcal{U}_x^{(i)}$. Similarly, they can be extended uniquely along any curve. The extension along a loop based at q produces a permutation of subspaces $\mathcal{U}_q^{1,i}, ..., \mathcal{U}_q^{l_i,i}$, which depends only on the homotopy class of the loop. Thus we obtain a natural homomorphism $\rho_i : \pi_1(\mathcal{M}, q) \to \Sigma(l_i)$ from the fundamental group of $\mathcal{M}$ to the group of permutations of l_i symbols. If the homomorphism ρ_i is trivial, then $\mathcal{U}_q^{1,i}, ..., \mathcal{U}_q^{l_i,i}$ extend globally to continuous sub-bundles $\mathcal{U}^{1,i}, ..., \mathcal{U}^{l_i,i}$ over $\mathcal{M}$.

We consider the homomorphisms ρ_i for each $i = 1, ..., k-1$ and denote their direct product by $\rho : \pi_1(\mathcal{M}, q) \to \Sigma(l_1) \times ... \times \Sigma(l_{k-1})$. If ρ is non-trivial we pass to a finite cover as follows. The kernel $H_q = \ker \rho$ is a normal subgroup of finite index in $\pi_1(\mathcal{M}, q)$. Hence there exists a finite cover $p : \tilde{\mathcal{M}} \to \mathcal{M}$ such that $p_*(\pi_1(\tilde{\mathcal{M}}, \tilde{q})) = H_q$ for any $\tilde{q} \in p^{-1}(q)$. By taking the pullback under p we obtain the bundle $\tilde{\mathcal{E}}$ over $\tilde{\mathcal{M}}$ and for each $i = 1, ..., k-1$ the corresponding union of subspaces $\tilde{\mathcal{U}}_{\tilde{x}}^{(i)}$ in $\tilde{\mathcal{E}}_{\tilde{x}}$. Fix some $\tilde{q} \in p^{-1}(q)$. We claim that for each i the subspaces in $\tilde{\mathcal{U}}_{\tilde{q}}^{(i)}$ extend to continuous n_i-dimensional sub-bundles $\tilde{\mathcal{U}}^{1,i}, ..., \tilde{\mathcal{U}}^{l_i,i}$ of $\tilde{\mathcal{E}}$ so that $\tilde{\mathcal{U}}_{\tilde{x}}^{1,i} \cup ... \cup \tilde{\mathcal{U}}_{\tilde{x}}^{l_i,i} = \tilde{\mathcal{U}}_{\tilde{x}}^{(i)}$ for all $\tilde{x} \in \tilde{\mathcal{M}}$. It suffices to check that any loop $\tilde{\gamma} \in \pi_1(\tilde{\mathcal{M}}, \tilde{q})$ induces trivial permutations.

Indeed, $p \circ \tilde\gamma \in H_q$ by the construction of $\tilde{\mathcal{M}}$, and $\tilde\gamma$ and $p \circ \tilde\gamma$ induce the same permutations since the extension along $\tilde\gamma$ projects to that along $p \circ \tilde\gamma$.

Now we lift f to the cover $\tilde{\mathcal{M}}$. Fix any $\tilde q \in p^{-1}(q)$ and choose any $\tilde x \in p^{-1}(f(q))$.

$$\begin{array}{ccc} \tilde{\mathcal{M}},\, \tilde q & \xrightarrow{\tilde f} & \tilde{\mathcal{M}},\, \tilde x \\ p\downarrow & & p\downarrow \\ \mathcal{M},\, q & \xrightarrow{f} & \mathcal{M},\, f(q) \end{array} \qquad\qquad \begin{array}{ccc} \pi_1(\tilde{\mathcal{M}},\, \tilde q) & & \pi_1(\tilde{\mathcal{M}}, \tilde x) \\ p_*\downarrow & & p_*\downarrow \\ \pi_1(\mathcal{M}, q) & \xrightarrow{f_*} & \pi_1(\mathcal{M}, f(q)) \end{array}$$

A necessary and sufficient condition for existence of $\tilde f : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}$ satisfying $\tilde f(\tilde q) = \tilde x$ and covering $f \circ p$ is the inclusion $(f \circ p)_*(\pi_1(\tilde{\mathcal{M}}, \tilde q)) \subset p_*(\pi_1(\tilde{\mathcal{M}}, \tilde x))$. We will show the equality. We recall that $p_*(\pi_1(\tilde{\mathcal{M}}, \tilde q)) = H_q$ by the construction. It follows that $p_*(\pi_1(\tilde{\mathcal{M}}, \tilde x))$ is the subgroup $H_{f(q)}$ of $\pi_1(\mathcal{M}, f(q))$ that consists of all loops inducing trivial permutations of the subspaces at $f(q)$. Indeed, for any natural isomorphism $i_s : \pi_1(\mathcal{M}, f(q)) \to \pi_1(\mathcal{M}, q)$ given by a path s from q to $f(q)$ it is easily seen that $i_s(H_{f(q)}) = H_q$, and for any cover $i_s(p_*(\pi_1(\tilde{\mathcal{M}}, \tilde x))) = p_*(\pi_1(\tilde{\mathcal{M}}, \tilde q))$ since the latter is normal. Finally, $f_*(H_q) = H_{f(q)}$ since for any loop $\gamma \in \pi_1(\mathcal{M}, q)$ the cocycle F gives a homeomorphism between the restrictions of $\mathcal{E}$ to γ and to $f \circ \gamma$, which maps $\mathcal{U}^{(i)}$ to $\mathcal{U}^{(i)}$ and hence preserves the type of the induced permutation. We conclude that $(f \circ p)_*(\pi_1(\tilde{\mathcal{M}}, \tilde q)) = p_*(\pi_1(\tilde{\mathcal{M}}, \tilde x))$ and thus the lift $\tilde f$ exists.

We note that the manifold $\tilde{\mathcal{M}}$ is compact and connected, and the lift $\tilde f$ satisfies our standing assumptions. Indeed, since the projection p is a local diffeomorphism, the invariant volume μ lifts to an invariant volume $\tilde\mu$, and the partially hyperbolic splitting for f lifts to the one for $\tilde f$. Moreover, $\tilde f$ satisfies the same bunching and its stable/unstable foliations project to those of f. It follows that $\tilde f$ is accessible. Indeed, by compactness and connectedness, it suffices to show that any point $\tilde q \in \tilde{\mathcal{M}}$ has a neighborhood whose any point $\tilde y$ can be connected to $\tilde q$ by an su-path. Let $q = p(\tilde q)$ and $y = p(\tilde y)$. By [W, Lemma 4.4] q can be connected to any sufficiently close y by an su-path arbitrarily close to a certain contractible su-path from q to q. The lift of such a path to $\tilde{\mathcal{M}}$ gives an su-paths connecting $\tilde q$ and $\tilde y$.

We denote by $\tilde F : \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$ the unique lift of the cocycle F to the cocycle over $\tilde f$. We note that $\tilde{\mathcal{E}}$ and $\tilde F$ are β-Hölder, and $\tilde F$ is fiber-bunched. We also lift the matrix functions A and C to $\tilde{\mathcal{M}}$, $\tilde A(\tilde x) = A(p(\tilde x))$ and $\tilde C(\tilde x) = C(p(\tilde x))$, and note that $\tilde A(\tilde x) = \tilde C^{-1}(\tilde f(\tilde x))\tilde F(\tilde x)\, \tilde C(\tilde x)$ for $\tilde\mu$-a.e. $\tilde x$, where $\tilde\mu$ is the lift of μ.

For each i the cocycle $\tilde F$ preserves the union $\tilde{\mathcal{U}}^{(i)}$ of sub-bundles $\tilde{\mathcal{U}}^{1,i}, ..., \tilde{\mathcal{U}}^{l_i,i}$. Since the sub-bundles are continuous, the permutation of their order induced by $\tilde F_{\tilde x}$ is continuous in $\tilde x$ and hence is constant on $\tilde{\mathcal{M}}$. Hence there exists N such that the cocycle $\hat F = \tilde F^N$ preserves every sub-bundle $\tilde{\mathcal{U}}^{j,i}$, $i = 1, ..., k-1$, $j = 1, ..., l_i$. Moreover, these sub-bundles can be arranged into flags $\tilde{\mathcal{W}}^1, ..., \tilde{\mathcal{W}}^l$ which are mapped by $\tilde C^{-1}(\tilde x)$ to the flags $W^1, ..., W^l$ in $\mathbb{R}^d$ for $\tilde\mu$-a.e. $\tilde x$ up to a permutation which depends on $\tilde x$. (We can not expect this permutation to be constant a.e. since modifying C on a set of positive measure by an element of G gives a different version of A satisfying the conclusion of the Amenable Reduction Theorem.) Indeed, for each i and for $\tilde\mu$-a.e. $\tilde x$ the isomorphism $\tilde C(\tilde x)$ maps $U^{(i)}$ to $\tilde{\mathcal U}^{(i)}_{\tilde x}$ and thus marks the inclusions between subspaces $\tilde{\mathcal U}^{1,i-1}_{\tilde x}, ..., \tilde{\mathcal U}^{l_{i-1},i-1}_{\tilde x}$ and $\tilde{\mathcal U}^{1,i}_{\tilde x}, ..., \tilde{\mathcal U}^{l_i,i}_{\tilde x}$ that correspond to the inclusions in the flags $W^1, ..., W^l$. We can view these inclusions as a measurable function ψ_i from $\tilde{\mathcal M}$ to the set of binary relations between the sets $\{1, ..., l_{i-1}\}$ and $\{1, ..., l_i\}$. We note that the cocycle $\hat F$ is conjugate by $\tilde C$ to the matrix cocycle $\hat A = \tilde A^N$ with values in the same group G. All elements of G permute the flags $W^1, ..., W^l$ and thus preserve the corresponding binary relations. Since $\hat F$ also preserves all sub-bundles $\tilde{\mathcal U}^{j,i}$, we conclude ψ_i is invariant under $\tilde f^N$ and hence is constant $\tilde\mu$-a.e. by ergodicity. The inclusions given by the functions ψ_i arrange the sub-bundles $\tilde{\mathcal U}^{j,i}$ into desired flags $\tilde{\mathcal W}^1, ..., \tilde{\mathcal W}^l$. We fix one these flags and denote it

$$\tilde{\mathcal W} = \left\{ \{0\} = \tilde{\mathcal E}^0 \subset \tilde{\mathcal E}^1 \subset \cdots \subset \tilde{\mathcal E}^k = \tilde{\mathcal E} \right\}.$$

It remains to show that each factor bundle $\tilde{\mathcal E}^i/\tilde{\mathcal E}^{i-1}$, $i = 1, ..., k$, has a continuous conformal structure invariant under the factor cocycle induced by $\hat F$.

Recall that any matrix A in G_0 has the block-triangular form (4.6) and hence preserves the subspaces V^i and V^{i-1}. Its factor map on $V^i/V^{i-1} = \mathbb R^{d_i}$ is given by the block A_i and thus preserves the standard conformal structure σ on $\mathbb R^{d_i}$. The orbit of σ under the flag stabilizer G_* is a finite set $\mathcal O$ in the space of conformal structures on $\mathbb R^{d_i}$. Since $\mathcal O$ is invariant under G_*, so is the smallest ball containing $\mathcal O$, which is unique in the space of nonpositive curvature. The center of this ball is a conformal structure σ_* on $V^i/V^{i-1} = \mathbb R^{d_i}$ preserved by the factor action of any matrix in G_*. Pushing σ_* by the action of G we obtain conformal structures σ_j on the factors $V^{j,i}/V^{j,i-1}$ of the flags W^j, $j = 1, ..., l$. It follows that if $g \in G$ maps W^j to $W^{j'}$ then its factor map $\bar g : V^{j,i}/V^{j,i-1} \to V^{j',i}/V^{j',i-1}$ takes σ_j to $\sigma_{j'}$. By the construction of the flag $\tilde{\mathcal W}$, for $\tilde\mu$-a.e. $\tilde x$ there is $j = j(x)$ such that $\tilde{\mathcal E}^i_{\tilde x} = \tilde C(\tilde x)V^{j,i}$ and $\tilde{\mathcal E}^{i-1}_{\tilde x} = \tilde C(\tilde x)V^{j,i-1}$. Pushing σ_j by $\tilde C(\tilde x)$ we obtain a measurable conformal structure τ on $\tilde{\mathcal E}^i/\tilde{\mathcal E}^{i-1}$. Since $\hat F$ is conjugate by $\tilde C$ to $\hat A$, which takes values in G, we conclude that τ is invariant under the factor of $\hat F$ on $\tilde{\mathcal E}^i/\tilde{\mathcal E}^{i-1}$. The stable and unstable holonomies for the fiber-bunched cocycle $\tilde F$ give the continuous holonomies for $\hat F$ and for its factor on $\tilde{\mathcal E}^i/\tilde{\mathcal E}^{i-1}$. Using Proposition 4.4 and Theorem 3.1 as before we conclude that τ coincides $\tilde\mu$-a.e. with a continuous conformal structure on $\tilde{\mathcal E}^i/\tilde{\mathcal E}^{i-1}$ invariant under the factor cocycle of $\hat F$. □

Remark 4.8. Suppose that there exist d continuous vector fields such that $\tilde{\mathcal E}^i$ is spanned by the first $d_1 + \cdots + d_i$ of them. Then the theorem implies that $\hat F$ is continuously cohomologous to a cocycle with values in a standard maximal amenable subgroup of $GL(d, \mathbb R)$ given by (4.6). Indeed, since $\tilde{\mathcal E} \approx \mathcal M \times \mathbb R^d$ we can view the cocycle as a function $\hat{F} : \mathcal{M} \to GL(d, \mathbb{R})$, moreover, the vector fields give a continuous coordinate change $\bar{C} : \mathcal{M} \to GL(d, \mathbb{R})$ such that $\bar{A}(x) = \bar{C}^{-1}(fx)\hat{F}(x)\,\bar{C}(x)$ has a block triangular form. Each diagonal block $\bar{A}_i$ corresponds to the factor cocycle on $\mathcal{E}^i/\mathcal{E}^{i-1}$ and thus preserves a continuous conformal structure τ_i on $\mathbb{R}^{d_i}$, i.e. $\bar{A}_i(x)(\tau_i(x)) = \tau_i(\tilde{f}x)$. To make the diagonal blocks conformal we change the coordinates in $\mathbb{R}^{d_i}$ by the unique positive square root of the matrix of $\tau_i(x)$.

4.6. **Example.** We give an example of an analytic cocycle F on $\mathcal{E} = \mathbb{T}^2 \times \mathbb{R}^2$ over an Anosov automorphism f of $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ so that F is fiber bunched and has only one Lyapunov exponent with respect to the Haar measure μ, but has no invariant μ-measurable sub-bundles or conformal structures.

We view cocycles as $GL(2, \mathbb{R})$-valued functions. We construct F using its 4-cover $\bar{F}$ on $\bar{\mathbb{T}}^2 = \mathbb{R}^2/(4\mathbb{Z} \times \mathbb{Z})$, which is conjugate to a diagonal cocycle $\bar{A}$. We define

$$\bar{A}(x) = \begin{bmatrix} a(x) & 0 \\ 0 & b(x) \end{bmatrix}, \quad \text{where} \ \ a(x) = 1 + \epsilon\cos(\pi x_1), \ \ b(x) = 1 - \epsilon\cos(\pi x_1);$$

$$\bar{C}(x) = \begin{bmatrix} \cos(\frac{\pi}{2}x_1) & -\sin(\frac{\pi}{2}x_1) \\ \sin(\frac{\pi}{2}x_1) & \cos(\frac{\pi}{2}x_1) \end{bmatrix} = R\left(\frac{\pi}{2}x_1\right), \quad \text{the rotation by } \ \frac{\pi}{2}x_1.$$

Both $\bar{C}$ and $\bar{A}$ are well-defined and analytic on $\bar{\mathbb{T}}^2$. We choose a hyperbolic matrix in $SL(2, \mathbb{Z})$ congruent to the identity modulo 4, e.g. $\left[\begin{smallmatrix} 41 & 32 \\ 32 & 25 \end{smallmatrix}\right]$, and let f and $\bar{f}$ be the induced automorphisms of $\mathbb{T}^2$ and $\bar{\mathbb{T}}^2$, respectively. Then we define the function

$$\bar{F}(x) = \bar{C}(\bar{f}x)\,\bar{A}(x)\,\bar{C}(x)^{-1} = \bar{C}(\bar{f}x)\,\bar{C}(x)^{-1} \cdot \bar{C}(x)\,\bar{A}(x)\,\bar{C}(x)^{-1}. \tag{4.9}$$

The term $\bar{C}(\bar{f}x)\,\bar{C}(x)^{-1}$ is the rotation $R\left(\frac{\pi}{2}((\bar{f}x)_1 - x_1)\right)$ and hence is 1-periodic in both x_1 and x_2 by the assumption on $\bar{f}$. A direct calculation shows that

$$\bar{C}(x)\,\bar{A}(x)\,\bar{C}(x)^{-1} = \frac{1}{2}\begin{bmatrix} 2 + \epsilon(1 + \cos(2\pi x_1)) & \epsilon\sin(2\pi x_1) \\ \epsilon\sin(2\pi x_1) & 2 - \epsilon(1 + \cos(2\pi x_1)) \end{bmatrix}.$$

Therefore $\bar{F}$ is 1-periodic in both x_1 and x_2, and it projects to an analytic function F on $\mathbb{T}^2$. For small ϵ, $\bar{F}$ and F are close to orthogonal, hence are fiber bunched.

Since $a(0) \neq b(0)$, the functions $a(x)$ and $b(x)$ are not cohomologous, and hence the coordinate line bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ are the only invariant sub-bundles for $\bar{A}$ measurable with respect to $\bar{\mu}$ [S, Lemma 7.1]. It is easy to see that $\bar{A}$ has one Lyapunov exponent

$$\lambda = \lim_{n\to\infty} \frac{\log(a(x)\dots a(f^{n-1}x))}{n} = \int_{\bar{\mathbb{T}}^2} \log a(x)\,d\bar{\mu} = \int_{\bar{\mathbb{T}}^2} \log b(x)\,d\bar{\mu} \quad \text{for } \bar{\mu}\text{-a.e. } x.$$

Since $\bar{F}$ is conjugate by $\bar{C}$ to $\bar{A}$, it also has one Lyapunov exponent with respect to $\bar{\mu}$, and hence so does F. However, F has two exponents at $0 = f(0)$, $\log(1+\epsilon)$ and $\log(1-\epsilon)$, so it cannot preserve a conformal structure. Also, $\bar{F}$ preserves exactly two sub-bundles $\bar{U}^i = \bar{C}\,\mathcal{E}_i$. Their projections to $\mathbb{T}^2$ are not sub-bundles, as they "twist together into a single object". (On the intermediate cover $\tilde{\mathbb{T}}^2 = \mathbb{R}^2/(2\mathbb{Z} \times \mathbb{Z})$ this object splits into two non-orientable invariant sub-bundles, illustrating the lift in the proof of Theorem 3.4.) We conclude that F has no invariant sub-bundle, since lifting one to $\bar{T}^2$ would give an invariant sub-bundle for $\bar{F}$ different from $\bar{U}^1$ and $\bar{U}^2$.

4.7. **Subadditive sequences of functions.** Proposition 4.9 plays a key role in the proof of Corollary 3.6. It removes extra assumptions from [R, Proposition 3.5], which was similar to a result in [Sch], but proved to be more useful for many applications.

Let f be a homeomorphism of a compact metric space X. A sequence of continuous functions $a_n : X \to \mathbb{R}$ is called *subadditive* if

$$a_{n+k}(x) \le a_k(x) + a_n(f^k x) \quad \text{for all } x \in X \text{ and } n, k \in \mathbb{N}. \tag{4.10}$$

For any Borel probability measure μ on X we denote $a_n(\mu) = \int_X a_n d\mu$. If μ if f-invariant, (4.10) implies that $a_{n+k}(\mu) \le a_n(\mu) + a_k(\mu)$, i.e. the sequence of real numbers $\{a_n(\mu)\}$ is subadditive. It is well known that for such a sequence the following limit exists:

$$\chi(\mu) := \lim_{n\to\infty} \frac{a_n(\mu)}{n} = \inf_{n\in\mathbb{N}} \frac{a_n(\mu)}{n}.$$

Also, by the Subaddititive Ergodic Theorem, if μ is ergodic then

$$\lim_{n\to\infty} \frac{a_n(x)}{n} = \chi(\mu) \quad \text{for } \mu\text{-almost all } x \in X. \tag{4.11}$$

Proposition 4.9. *Let f be a homeomorphism of a compact metric space X and $a_n : X \to \mathbb{R}$ be subadditive sequence of continuous functions. If $\chi(\mu) < 0$ for every ergodic invariant Borel probability measure μ for f, then there exists N such that $a_N(x) < 0$ for all $x \in X$.*

Proof. We denote by $\mathcal{M}$ the set of f-invariant Borel probability measures on X. First we note that, by the Ergodic Decomposition, if $\chi(\mu) < 0$ for every *ergodic* $\mu \in \mathcal{M}$, then the same holds for every $\mu \in \mathcal{M}$.

First we show that there exists K such that $a_K(\mu) < c < 0$ for all $\mu \in \mathcal{M}$. Since $\chi(\mu) < 0$ there exists n_μ and c_μ such that $a_{n_\mu}(\mu) < 2c_\mu < 0$. Since a_n are continuous, for every $\mu \in \mathcal{M}$ there is a neighborhood V_μ in the weak* topology such that $a_{n_\mu}(\nu) < c_\mu$ for every $\nu \in V_\mu$. We choose a finite cover $\{V_{\mu_i},\ i \in I\}$ of $\mathcal{M}$ and set $R = \max_I n_i$ and $c = \max_I c_i$. Take any $\mu \in \mathcal{M}$ and let i be such that $\mu \in V_{\mu_i}$. For any K we can write $K = kn_i + r$, where $r < n_i \le R$, and by the subadditivity we get

$$a_K(\mu) \le k a_{n_i}(\mu) + a_r(\mu) < kc + a_r(\mu).$$

Since a_r are uniformly bounded for $r < R$, we conclude that $a_K(\mu) < c < 0$ provided that K, and hence k, are sufficiently large.

Lemma 4.10. *Suppose that for a continuous function $\phi : X \to \mathbb{R}$, $\phi(\mu) < c$ for all $\mu \in \mathcal{M}$. Then there exists n_0 such that $\frac{1}{n}\sum_{i=0}^{n-1} \phi(f^i x) < c$ for all $x \in X$ and $n \ge n_0$.*

Proof. Suppose on the contrary that there exit sequences $x_j \in X$ and $n_j \to \infty$ such that $S_j = \frac{1}{n_j}\sum_{i=0}^{n_j-1} \phi(f^i x) \ge c$. Note that $S_j = \psi(\mu_j)$, where $\mu_j = \frac{1}{n_j}\sum_{i=0}^{n_j-1} \delta(f^i x_j)$

is a probability measure. Using compactness of the set of probability measures on X we may assume, by passing to a subsequence if necessary, that μ_j weak* converges to a probability measure μ. Since the total variation norm $\|f_*\mu_j - \mu_j\| \le \frac{2}{n_j}$ it follows that the limit μ is f-invariant. On the other hand $\psi(\mu) = \lim \psi(\mu_j) \ge c$, which contradicts the assumption. □

Applying the lemma to a function a_K satisfying $a_K(\mu) < c < 0$ we conclude that there is n_0 such that $\sum_{i=0}^{n-1} a_K(f^i x) < cn$ for all $n \ge n_0$ and $x \in X$. Let $n = Km \ge n_0$. Using subadditivity repeatedly, for $i = 0, \dots, K-1$ we obtain

$$\begin{aligned} a_n(f^i x) &\le a_K(f^i x) + a_{n-K}(f^{K+i} x) \le \dots \\ &\le a_K(f^i x) + a_K(f^{K+i} x) + \dots + a_K\left(f^{(m-1)K+i} x\right). \end{aligned}$$

Adding these K inequalities, we get

$$a_n(x) + a_n(fx) + \dots + a_n(f^{K-1}x) \;\le\; \sum_{i=0}^{n-1} a_K(f^i x) \le cn.$$

Let $N = n + K$. For $i = 0, \dots, K-1$, we obtain

$$a_N(x) \le a_i(x) + a_{n+K-i}(f^i x) \le a_i(x) + a_n(f^i x) + a_{K-i}(f^{n+i} x) =: a_n(f^i x) + \Delta_i(x),$$

where we set $a_0(x) = 0$. Let $M = \max\{\, |\Delta_i(x)| : \ 0 \le i \le K-1, \ x \in \mathcal{M}\}$. Adding the inequalities, we get

$$K \cdot a_N(x) \le a_n(x) + a_n(fx) + \dots + a_n(f^{K-1}x) + KM \le cn + KM$$

and hence $a_N(x) \le cn/K + M$. Since $c < 0$, taking $m = n/K$ sufficiently large, we can ensure that $a_N(x) < 0$ for all x. □

4.8. **Proof of Corollary 3.6.** The fiber bunching assumption is used in the proofs of Theorems 3.1, 3.3, and 3.4 only to obtain the stable and unstable holonomies for F. Thus it suffices to show that the assumption in the corollary also ensures their existence. Applying Proposition 4.11 below with $\xi = 0$ we obtain that for every $\epsilon > 0$ there exists C_ϵ such that

$$\|F_x^n\| \cdot \|(F_x^n)^{-1}\| \le C_\epsilon e^{\epsilon |n|} \quad \text{for all } x \in \mathcal{M} \text{ and } n \in \mathbb{Z}. \tag{4.12}$$

We consider ϵ such that $e^\epsilon < \min \nu(x)^{-\beta}$. Then (4.12) implies that for all sufficiently large n the cocycle F^n over f^n is fiber bunched and, by Proposition 4.2, has the stable and unstable holonomies. The holonomies for both F^N and F^{N+1} are also holonomies for $F^{N(N+1)}$ and hence coincide by uniqueness, Proposition 4.2 (c). This easily implies that they are also holonomies for F. We only need to check Definition 4.1(iii) for $n = 1$ which follows from that for $n = N+1$ at x and for $n = N$ at fx:

$$\begin{aligned} H^s_{xy} &= (F_y^{N+1})^{-1} \circ H^s_{f^{N+1}x\, f^{N+1}y} \circ F_x^{N+1} = \\ &= (F_y)^{-1} \circ (F_{fy}^N)^{-1} \circ H^s_{f^{N+1}x\, f^{N+1}y} \circ F^N_{fx} \circ F_x = (F_y)^{-1} \circ \ H^s_{fx\, fy} \circ F_x. \end{aligned}$$

Proposition 4.11. *Suppose that there exists $\xi \ge 0$ such that $\lambda_+(F,\mu)-\lambda_-(F,\mu) \le \xi$ for every ergodic f-invariant measure μ. Then for any $\epsilon > 0$ there exists C_ϵ such that*

$$K_F(x,n) = \|F_x^n\| \cdot \|(F_x^n)^{-1}\| \le C_\epsilon e^{(\xi+\epsilon)|n|} \quad \textit{for all } x \in \mathcal{M} \textit{ and } n \in \mathbb{Z}. \tag{4.13}$$

Proof. The proof is similar to that of [KS10, Proposition 2.1] but simpler due to the use of Proposition 4.9. For a given $\epsilon > 0$ we apply Proposition 4.9 to the functions

$$a_n(x) = \log K_F(x,n) - (\xi+\epsilon)n, \quad n \in \mathbb{N}.$$

It is easy to see from the definition of the quasiconformal distortion that

$$K(x,n+k) \le K(x,k) \cdot K(f^k x, n) \quad \text{for every } x \in \mathcal{M} \text{ and } n,k \ge 0. \tag{4.14}$$

It follows that the sequence of functions $\{a_n\}$ is subadditive.

Let μ be an ergodic f-invariant measure. As in (4.5) we obtain that

$$\lim_{n\to\infty} n^{-1} \log K(x,n) = \lambda_+(F,\mu) - \lambda_-(F,\mu) \le \xi \quad \text{for } \mu\text{-a.e. } x.$$

It follows that $\lim_{n\to\infty} n^{-1} a_n(x) \le -\epsilon < 0$ for μ-a.e. x, and (4.11) implies that $\chi(\mu) < 0$ for the sequence $\{a_n\}$. Hence there exists N_ϵ such that $a_{N_\epsilon}(x) < 0$, i.e. $K(x,N_\epsilon) \le e^{(\xi+\epsilon)N_\epsilon}$ for all $x \in \mathcal{M}$. The proposition follows for $n > 0$ from (4.14) by choosing C_ϵ sufficiently large, and for $n < 0$ since $K(x,n) = K(f^n x, -n)$. □

4.9. **Proof of Corollary 3.7.** Let τ be the continuous F-invariant conformal structure obtained in Theorem 3.1. It follows as in the proof of Proposition 4.4 that $\tau(y) = H^s_{xy}(\tau(x))$ for all $x \in \mathcal{M}$ and $y \in W^s_{loc}(x)$. By Proposition 4.2, $\|H^s_{xy} - I_{xy}\| \le C \operatorname{dist}(x,y)^\beta$, and hence by Lemma 4.5 $\operatorname{dist}(I_{xy}(\tau(x)), \tau(y)) \le C' \operatorname{dist}(x,y)^\beta$. The same holds for all $y \in W^u_{loc}(x)$. Now local α-Hölder accessibility implies that τ is $\alpha\beta$-Hölder on $\mathcal{M}$.

In the proof of Theorem 3.3 we established that the sub-bundle is invariant under the stable and unstable holonomies. Thus by the same reason it is β-Hölder along W^u and W^s, and $\alpha\beta$-Hölder on $\mathcal{M}$. Since the lift $\tilde{f}$ in Theorem 3.4 is also locally α-Hölder accessible, we obtain the same regularity for the sub-bundles $\tilde{\mathcal{E}}^i$ and the factor bundles $\tilde{\mathcal{E}}^i/\tilde{\mathcal{E}}^{i-1}$. In particular, the holonomies for the induced cocycle on $\tilde{\mathcal{E}}^i/\tilde{\mathcal{E}}^{i-1}$ are β-Hölder along $\tilde{W}^u$ and $\tilde{W}^s$. As above, the conformal structure on $\tilde{\mathcal{E}}^i/\tilde{\mathcal{E}}^{i-1}$ is invariant under these holonomies and hence it is $\alpha\beta$-Hölder on $\tilde{\mathcal{M}}$. □

4.10. **Proof of Corollary 3.8.** The difference in the proof for the partially hyperbolic and Anosov case is in obtaining global continuity on $\mathcal{M}$ from that along the stable and unstable foliations, and this argument is more direct in the Anosov case.

Theorem 3.1 for Anosov case follows from Proposition 4.2 (a) and [KS10, Proposition 2.3]. Alternatively, as in Theorem 3.1, we obtain that the conformal structure τ is essentially β-Hölder along W^s and W^u and then conclude that it is essentially β-Hölder on $\mathcal{M}$ using 1-Hölder accessibility and the local product structure of the measure, as in the end of the proof of [KS10, Proposition 2.3]. In the proof of Theorem 3.3 we only need to replace the reference [ASV, Theorem C] by [AV, Theorem

D] and then conclude that the invariant distribution is β-Hölder by Corollary 3.7. Using these results, Corollary 3.2 and Theorem 3.4 can be obtained as in this paper.

The prove the last statement of the corollary we recall that for Hölder continuous linear cocycles over hyperbolic systems, the Lyapunov exponents of any ergodic invariant measure can be approximated by Lyapunov exponents of periodic measures [K, Theorem 1.4]. Hence we obtain $\lambda_+(F,\eta) = \lambda_-(F,\eta)$ for every ergodic f-invariant measure η and the argument in the proof of Corollary 3.6 applies.

4.11. **Proof of Theorem 3.9.** By Corollary 3.8 the conclusion of Theorem 3.4 holds, moreover, the invariant sub-bundles and conformal structures are β-Hölder.

We use notations from the proof of Theorem 3.4. Recall that $\tilde{F} : \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$ is the lift of F to the cocycle over $\tilde{f} : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}$, and that the cocycle $\hat{F} = \tilde{F}^N$ preserves the flags $\tilde{\mathcal{W}}^1, ..., \tilde{\mathcal{W}}^l$. We denote these β-Hölder flags by

$$\tilde{\mathcal{W}}^j = \Big\{ \{0\} = \tilde{\mathcal{E}}^{j,0} \subset \tilde{\mathcal{E}}^{j,1} \subset \cdots \subset \tilde{\mathcal{E}}^{j,k} = \tilde{\mathcal{E}} \Big\}, \qquad j = 1, ..., l.$$

Now we define sub-bundles $\hat{\mathcal{E}}^i = \sum_{j=1}^{l} \tilde{\mathcal{E}}^{j,i}$, where the sum may be not direct, which form a new β-Hölder flag, where the inclusions may be not strict. By the construction of $\tilde{\mathcal{W}}^j$, this flag projects to a β-Hölder F-invariant flag $\{0\} = \mathcal{E}^0 \subset \mathcal{E}^1 \subseteq \mathcal{E}^2 \subseteq \cdots \subseteq \mathcal{E}^k = \mathcal{E}$. Eliminating unnecessary sub-bundles in case of equalities we obtain the desired flag (3.5). We will show below that for each $i = 1, ..., k$ the factor bundle $\hat{\mathcal{E}}^i / \hat{\mathcal{E}}^{i-1}$ has a continuous conformal structure invariant under the factor of $\hat{F}$. This implies uniform quasiconformality of the factor cocycle $F^{(i)}$ induced by F on $\mathcal{E}^i/\mathcal{E}^{i-1}$ and, by Corollaries 3.2 and 3.8, existence of a β-Hölder conformal structure τ_i on $\mathcal{E}^i/\mathcal{E}^{i-1}$ invariant under $F^{(i)}$, $i = 1, ..., k$. Then the proof is completed as follows.

We normalize the conformal structure τ_i to get a β-Hölder Riemannian metric $\|.\|_i'$ on $\mathcal{E}^i/\mathcal{E}^{i-1}$, $i = 1, ..., k$, with respect to which $F^{(i)}$ is conformal. We denote by $a_i(x)$ the scaling coefficients with respect to these norms:

$$\|F^{(i)}(v)\|_i' = a_i(x)\|v\|_i' \quad \text{ for all } v \in \mathcal{E}^i/\mathcal{E}^{i-1}\,(x).$$

These are positive β-Hölder functions, and for each periodic point $p = f^n p$ the product $a_i(f^{n-1}p) \cdots a_i(fp)\, a_i(p)$ is the same for all i. Indeed, the matrix of F_p^n is block triangular in any basis of $\mathcal{E}_p$ appropriate for the flag, and the above product is simply the modulus of the eigenvalues of the block corresponding to $\mathcal{E}^i/\mathcal{E}^{i-1}$; all these moduli are equal since F_p^n has only one Lyapunov exponent. Now by the Livšic theorem [L],[KtH, Theorem 19.2.1] this implies that the functions a_i are cohomologous, more precisely there exist positive β-Hölder functions ψ_i such that for all x

$$a_i(x) = a_1(x)\psi_i(fx)\psi_i(x)^{-1}, \quad i = 2, ..., k.$$

Choosing new metric $\|.\|_i = \psi_i^{-1}\|.\|_i'$, $i = 2, ..., k$, makes a_1 the scaling coefficient for all $F^{(i)}$. Hence the cocycle $a_1(x)^{-1}F(x)$ induces isometries on each $\mathcal{E}^i/\mathcal{E}^{i-1}$, $i = 1, ..., k$.

It remains to obtain conformal structures on $\hat{\mathcal{E}}^i/\hat{\mathcal{E}}^{i-1}$ invariant under the factors of $\hat{F}$. We fix $1 \le i \le k$. As we showed in the proof of Theorem 3.4, for each $j = 1, ..., l$

the factor of $\hat{F}$ on $\tilde{\mathcal{E}}^{j,i}/\tilde{\mathcal{E}}^{j,i-1}$ has an invariant conformal structure, which in our case is β-Hölder. We normalize these structures to obtain β-Hölder Riemannian metrics g_j. As in the argument above we can show that the scaling coefficients of $\hat{F}$ are cohomologous functions. Hence we may assume that the metrics g_j are normalized so that for some positive function φ the cocycle $\bar{F} = \varphi\hat{F}$ induces an isometry on $\tilde{\mathcal{E}}^{j,i}/\tilde{\mathcal{E}}^{j,i-1}$ for each $j = 1, ..., l$. To simplify notations we will also write $\bar{F}$ for its induced map on any factor bundle.

We fix j and consider $\bar{\mathcal{E}}^j = \tilde{\mathcal{E}}^{j,i}/(\tilde{\mathcal{E}}^{j,i} \cap \hat{\mathcal{E}}^{i-1})$ as a factor bundle of $\tilde{\mathcal{E}}^{j,i}/\tilde{\mathcal{E}}^{j,i-1}$. Since $\bar{F}$ is isometric on $\tilde{\mathcal{E}}^{j,i}/\tilde{\mathcal{E}}^{j,i-1}$, it preserves the orthogonal complement of $(\tilde{\mathcal{E}}^{j,i} \cap \hat{\mathcal{E}}^{i-1})/\tilde{\mathcal{E}}^{j,i-1}$ and the metric g_i restricted to it. This orthogonal complement is isomorphic to $\bar{\mathcal{E}}^j$, and thus we obtain an $\bar{F}$-invariant metric $\bar{g}_j$ on $\bar{\mathcal{E}}^j$. Now we view $\bar{\mathcal{E}}^j$ as a sub-bundle of $\hat{\mathcal{E}}^i/\hat{\mathcal{E}}^{i-1}$, so that $\hat{\mathcal{E}}^i/\hat{\mathcal{E}}^{i-1} = \sum_{j=1}^{l} \bar{\mathcal{E}}^j$, and combine the metrics $\bar{g}_j$ as follows. Let $\mathcal{U} = \bar{\mathcal{E}}^1 \cap \bar{\mathcal{E}}^2$ and $\mathcal{U}^\perp$ be its orthogonal complement in $\bar{\mathcal{E}}^2$. As before, $\bar{F}$ preserves $\mathcal{U}^\perp$ and the restriction of $\bar{g}_2$ to it. We combine g_1 and g_2 into $\bar{F}$-invariant the Riemannian metric on $\bar{\mathcal{E}}^1 + \bar{\mathcal{E}}^2 = \bar{\mathcal{E}}^1 \oplus \mathcal{U}^\perp$ by declaring the last two bundles orthogonal. Continuing this inductively we obtain a β-Hölder Riemannian metric on $\hat{\mathcal{E}}^i/\hat{\mathcal{E}}^{i-1}$ with respect to which $\bar{F}$ is isometric and $\hat{F}$ conformal.

4.12. **Proof of Theorem 3.10.** By Theorem 3.9 the cocycle $G(x) = \phi(x)F(x)$ induces isometries on each factor bundle $\mathcal{E}^i/\mathcal{E}^{i-1}$. Inductive application of the next proposition shows that $\|G^n(x)\| \le Dn^{k-1}$. Applying it to G^{-1} yields $\|G^{-n}(x)\| \le Dn^{k-1}$, and hence the quasiconformal distortion satisfies $K_G(x, n) \le Cn^{2(k-1)}$. Since $K_F(x, n) = K_G(x, n)$, the first part of the theorem follows.

If $\lambda_+(F, \mu_p) = \lambda_-(F, \mu_p) = 0$, then $\phi(x)$ is cohomologous to the constant 1 and, by rescaling the norm, we obtain that F itself induces isometries on each factor bundle $\mathcal{E}^i/\mathcal{E}^{i-1}$. Hence the second part also follows from the next proposition.

Proposition 4.12. *Let $F : \mathcal{E} \to \mathcal{E}$ be a continuous linear cocycle over f and let V be an F-invariant continuous sub-bundle. Suppose that the factor cocycle $\bar{F} : \mathcal{E}/V \to \mathcal{E}/V$ is an isometry and that for some C and k the restriction $F_V = F|_V$ satisfies $\|F_V^n(x)\| \le Cn^j$ for all x and $n \in \mathbb{N}$. Then there exists a constant D such that $\|F^n(x)\| \le Dn^{j+1}$ for all x and $n \in \mathbb{N}$.*

Proof. We denote by $P : \mathcal{E} \to \mathcal{E}/V$ the natural projection and by $\pi : \mathcal{E} \to V$ the orthogonal projection with respect to some Riemannian metric on $\mathcal{E}$. Then for any x the map $v \mapsto (P(v), \pi(v))$ identifies $\mathcal{E}(x)$ with $\mathcal{E}/V(x) \oplus V(x)$, and $\max\{\|(P(v)\|, \|\pi(v)\|\}$ gives a convenient continuous norm on $\mathcal{E}$. Since the linear map

$$\Delta'(x) = (\pi \circ F - F_V \circ \pi)(x) : \mathcal{E}(x) \to V(fx)$$

is identically zero on $V(x)$ we can write it as $\Delta'(x) = \Delta(x) \circ P$, where the linear map $\Delta(x) : \mathcal{E}/V(x) \to V(fx)$ depends continuously on x. Thus we have

$$\pi \circ F = F_V \circ \pi + \Delta \circ P, \quad P \circ F^n = \bar{F}^n \circ P, \qquad \text{and hence}$$

$$\begin{aligned}\pi \circ F^n &= (F_V \circ \pi + \Delta \circ P) \circ F^{n-1} = F_V \circ (\pi \circ F^{n-1}) + \Delta \circ \bar{F}^{n-1} \circ P = \\ &= F_V \circ ((F_V \circ \pi + \Delta \circ P) \circ F^{n-2}) + \Delta \circ \bar{F}^{n-1} \circ P = \\ &= F_V^2 \circ (\pi \circ F^{n-2}) + F_V \circ \Delta \circ \bar{F}^{n-2} \circ P + \Delta \circ \bar{F}^{n-1} \circ P = \cdots = \\ &= F_V^n \circ \pi + \sum_{i=0}^{n-1} F_V^{n-i-1} \circ \Delta \circ \bar{F}^i \circ P.\end{aligned}$$

Let K be such that $\|\Delta(x)\| \le K$ for all x. Since by the assumptions $\|F_V^i\| \le Cn^i$ and $\|\bar{F}^i\| = 1$, we can estimate

$$\|\pi \circ F^n(x)\| \le Cn^j + \sum_{i=0}^{n-1} C(n-i-1)^j \cdot K \le Cn^j + nCn^jK \le Dn^{j+1}$$

for some constant D independent of n and x. Since $\|P \circ F^n\| = \|\bar{F}^n \circ P\| \le 1$, we conclude that

$$\|F^n(x)\| = \max\{\|P \circ F^n(x)\|, \|\pi \circ F^n(x)\|\} \le Dn^{j+1}.$$

□

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