ON POINTWISE DIMENSION OF NON-HYPERBOLIC MEASURES

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ABSTRACT. We construct a diffeomorphism preserving a non-hyperbolic measure whose pointwise dimension does not exist almost everywhere. In one-dimensional case we also show that such diffeomorphisms are typical in certain situations.

1. INTRODUCTION

We consider an ergodic measure μ invariant under a diffeomorphism f of a compact Riemannian manifold $\mathcal{M}$. Such a measure μ is called **hyperbolic** if all its Lyapunov exponents are different from zero. The main goal of this paper is to show that hyperbolicity of a measure is essential for existence of its pointwise dimension.

We recall that the **pointwise dimension** at a point x of a Borel measure μ on a metric space X is defined as the following limit:

$$d_\mu(x) = \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r},$$

where $B(x,r)$ is a ball of radius r centered at $x \in X$. This limit does not exist in general. However the upper and lower pointwise dimensions $\overline{d}_\mu(x)$ and $\underline{d}_\mu(x)$ can be defined at any point x as corresponding upper and lower limits.

The study of pointwise dimension of hyperbolic measures in [3] has led to the problem known as the Eckmann-Ruelle conjecture. The complete affirmative solution of this problem was obtained by Barreira, Pesin and Schmeling:

Theorem ([2]). Let $f : \mathcal{M} \to \mathcal{M}$ be a $C^{1+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold $\mathcal{M}$. If μ is a hyperbolic ergodic measure for f then the pointwise dimension of μ exists for μ-almost every $x \in \mathcal{M}$ and is constant.

One may ask what happens if the requirement that μ is hyperbolic is omitted. The first result along this direction was obtained by Ledrappier and Misiurewicz in [7]. They constructed an example of a C^r-smooth map of an interval preserving an ergodic measure with zero Lyapunov exponent whose pointwise dimension does not exist almost everywhere. For the discussion of the above results see [8].

In this paper we consider circle diffeomorphisms with irrational rotation number which are known to be uniquely ergodic and have zero Lyapunov exponent.

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In Section 2 we prove genericity of circle diffeomorphisms f with irrational rotation number whose unique invariant measure μ_f has lower pointwise dimension 0 and upper pointwise dimension 1 for μ_f-almost every point in S^1. We also prove density of circle diffeomorphisms with irrational rotation number and given lower pointwise dimension of the unique invariant measure.

In Section 3 we show that circle homeomorphisms g with given upper and lower pointwise dimension of the unique invariant measure μ_g are dense in the set of all circle homeomorphisms with any given irrational rotation number.

In Section 4 we prove genericity of analytic circle diffeomorphism f with irrational rotation number whose unique invariant measure μ_f has lower pointwise dimension 0 and upper pointwise dimension 1 for μ_f-almost every point.

Let f be a circle diffeomorphism such that its unique invariant measure μ_f has lower pointwise dimension 0 and upper pointwise dimension 1 for μ_f-almost every point. Consider the direct product of a volume preserving Anosov diffeomorphism and f. It is easy to see that we obtain a partially hyperbolic diffeomorphism with only one zero Lyapunov exponent. The product measure is ergodic with respect to this diffeomorphism and its pointwise dimension does not exist almost everywhere. This shows that hyperbolicity of the measure is crucial in the Eckmann-Ruelle conjecture.

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2. Circle Diffeomorphisms

We adopt the following notation. Denote by $D_I^r \subset \mathrm{Diff}^r(S^1)$ the set of all C^r circle diffeomorphisms with irrational rotation number (see [6] for definition and properties of rotation number).

Let $Y^r \subset D_I^r$ be the set of all C^r circle diffeomorphisms f with irrational rotation number satisfying the following properties:

(1) $\underline{d}_\mu(x) = 0$ and $\overline{d}_\mu(x) = 1$ for μ-a.e. $x \in S^1$,
(2) $\dim_H \mu = \underline{\dim}_B \mu = 0$ and $\overline{\dim}_B \mu = 1$,

where μ is the invariant measure for f.

We recall the following definitions of Hausdorff, upper and lower box dimensions of a Borel probability measure μ:

$$\begin{aligned}
\dim_H \mu &= \inf\{\, \dim_H X :\ \mu(X) = 1 \,\},\\
\underline{\dim}_B \mu &= \lim_{\varepsilon\to 0} \inf\{\, \underline{\dim}_B X :\ \mu(X) > 1-\varepsilon \,\},\\
\overline{\dim}_B \mu &= \lim_{\varepsilon\to 0} \inf\{\, \overline{\dim}_B X :\ \mu(X) > 1-\varepsilon \,\}.
\end{aligned}$$

(See [4] for definition and properties of Hausdorff and box dimensions).

Our main results for circle diffeomorphisms are Theorem 2.1, Corollary 2.1, and Theorem 2.2.

Theorem 2.1. *For any* $0 \le r \le \infty$, Y^r *is a residual subset of both* D_I^r *and* $\overline{D_I^r}$ *(the closure of* D_I^r *in* C^r*-topology).*

Let D_τ^r be the set of all C^r circle diffeomorphisms with rotation number τ.

Corollary 2.1. *For any* $0 \le r \le \infty$, *there exists a set* $T^r \subset [0,1] \setminus \mathbb{Q}$ *which is a residual subset of* $[0,1]$ *such that for any* $\tau \in T^r$, $Y^r \cap D_\tau^r$ *is a residual subset of* D_τ^r.

Remark 2.1. Recall that a number τ is called Diophantine if it satisfies the following condition:

there exist $\delta > 0$ and $K > 0$ such that for any $p/q \in \mathbb{Q}$,

$$|\,\tau - p/q\,| > \frac{K}{|q|^{\,2+\delta}}.$$

Let f be a $C^{2+\varepsilon}$ circle diffeomorphism, where $\varepsilon > 0$, and its rotation number τ satisfy the Diophantine condition with some $K > 0$ and $0 < \delta < \varepsilon$. Then f is conjugate to the rotation by τ via a C^1 diffeomorphism (see [5]). This implies that the pointwise dimension of the invariant measure for f exists at every point $x \in S^1$ and is equal to 1.

Note that for any $\delta > 0$ the set of all numbers satisfying the Diophantine condition with some $K > 0$ has full Lebesgue measure. Therefore, the set T^r has zero Lebesgue measure for any $r > 2$. One can also show that $\dim_H T^r \le 2/r$ for any $2 < r < \infty$, and $\dim_H T^\infty = 0$.

The following theorem shows that any given number β, $0 < \beta < 1$, can be the value of the lower pointwise dimension of the invariant measure for a circle diffeomorphism.

Theorem 2.2. *For any given* $0 < \beta < 1$ *and* $0 \le r \le \infty$ *the set of all* C^r *circle diffeomorphisms* f *with irrational rotation number satisfying the following properties:*

(1) $\underline{d}_\mu(x) = \beta$ *and* $\overline{d}_\mu(x) = 1$ *for* μ*-a.e.* $x \in S^1$;

(2) $\dim_H \mu = \underline{\dim}_B \mu = \beta$ *and* $\overline{\dim}_B \mu = 1$,

is a dense subset of D_I^r.

Note that the set of diffeomorphisms described in Theorem 2.2 is not residual.

We begin with a construction of a uniquely ergodic circle diffeomorphism which is close to a given diffeomorphism and whose invariant measure μ does not have pointwise dimension almost everywhere. Our construction is closely related to the construction in [6] of circle diffeomorphisms conjugated to rotations via maps with specific degrees of regularity. The latter construction is based on a method developed by D. Anosov and A. Katok in [1] to construct examples of diffeomorphisms with specific ergodic properties.

Proposition 2.1. *Let $f_* : S^1 \to S^1$ be a C^∞ circle diffeomorphism such that $f_* = h_*^{-1} \circ R_{\tau_*} \circ h_*$, where h_* is a C^∞ circle diffeomorphism and R_{τ_*} is a circle rotation by τ_*.*

Then in any C^∞ neighborhood of f_* there exists a C^∞ diffeomorphism $f : S^1 \to S^1$ with irrational rotation number such that for its unique invariant measure μ, $\underline{d}_\mu(x) = 0$ and $\overline{d}_\mu(x) = 1$ for μ-a.e. $x \in S^1$.

Proof. The desired diffeomorphism f will be obtained as a limit of a sequence of diffeomorphisms $f_n = h_n^{-1} \circ R_{\tau_n} \circ h_n$, where $\tau_n = p_n/q_n$ is a rational number and h_n is a C^∞ diffeomorphism of S^1.

The sequences τ_n and h_n will be defined inductively as follows. We take $h_0 = h_*$ and a rational number τ_0 close to τ_*. Once $\tau_{n-1} = p_{n-1}/q_{n-1}$ and h_{n-1} are chosen we construct h_n as the composition $h_n = A_n \circ h_{n-1}$. The diffeomorphism A_n will be constructed in the form $A_n = Id + a_n$, where Id is the identity map and a_n is a $1/q_{n-1}$-periodic C^∞ function on S^1 such that a_n is zero in disjoint neighborhoods of points $k/q_{n-1}, \quad k = 1, ..., q_{n-1}$. A particular choice of A_n will be described later. Once A_n is constructed we choose τ_n in the form $\tau_n = \tau_{n-1} + (1/K_n q_{n-1})$, where K_n is an integral number. We choose K_n large enough as follows to ensure C^∞ convergence of diffeomorphisms f_n and C^0 convergence of diffeomorphisms h_n.

The C^0 distance between h_n and h_{n-1} (and between h_n^{-1} and h_{n-1}^{-1}) is bounded by $1/q_{n-1}$, the period of a_n. Therefore the sequence of diffeomorphisms h_n converges in C^0 topology to a homeomorphism $h = \lim_{n\to\infty} h_n$ if the sequence q_n grows sufficiently fast. This can be easily ensured by choosing K_n large enough.

Since $R_{p_{n-1}/q_{n-1}}$ and A_n^{-1} commute due to the form in which A_n is constructed we can rewrite f_n in the following way:

$$f_n = h_n^{-1} \circ R_{\tau_n} \circ h_n = h_{n-1}^{-1} \circ A_n^{-1} \circ R_{p_{n-1}/q_{n-1}} \circ R_{1/K_n q_{n-1}} \circ A_n \circ h_{n-1} =$$

$$= h_{n-1}^{-1} \circ R_{p_{n-1}/q_{n-1}} \circ A_n^{-1} \circ R_{1/K_n q_{n-1}} \circ A_n \circ h_{n-1}.$$

So we see that given a map A_n in the described form we can choose K_n so large that the map $A_n^{-1} \circ R_{1/K_n q_{n-1}} \circ A_n$ is close to Id in C^∞. It follows that we can make f_n be as close to f_{n-1} in C^∞ as we wish. This allows us to choose any A_n within described restrictions and then choose K_n so that the sequence f_n converges in C^∞ and its limit f is as close to f_0 as we wish. Taking τ_0 close to τ_* we can make f close to f_*.

Note that for the diffeomorphism f the rotation number $\tau = \lim_{n\to\infty} \tau_n$ is irrational once K_n grow to infinity. Indeed, suppose that $\tau = p/q \in \mathbb{Q}$. Then

$$\tau - \tau_n \,=\, p/q - p_n/q_n \,=\, \frac{pq_n - qp_n}{qq_n} \,\geq\, \frac{1}{qq_n}\,.$$

On the other hand,

$$\tau - \tau_n = \sum_{i=1}^{\infty} \frac{1}{q_n K_{n+1} \dots K_{n+i}} \le \frac{1}{q_n} \sum_{i=1}^{\infty} \frac{1}{K_{n+1}^i} = \frac{1}{q_n (K_{n+1}-1)} ,$$

which contradicts the previous estimate if n is sufficiently large.

We now specify the choice of A_n. Let μ be the invariant measure for f. We note that h is the distribution function of μ, i.e. $\mu([x_1, x_2)) = h(x_2) - h(x_1)$ for any interval $[x_1, x_2) \subset S^1$. Let $\Delta h(x, r) = h(x + r) - h(x - r)$. Then

$$\overline{d}_\mu(x) = \limsup_{r \to 0} \frac{\log \Delta h(x, r)}{\log r} \quad \text{and} \quad \underline{d}_\mu(x) = \liminf_{r \to 0} \frac{\log \Delta h(x, r)}{\log r}.$$

We think of A_n and a_n as C^∞ functions on the unit interval. Recall that a_n is periodic with period $s_n = 1/q_{n-1}$ and A_n is monotone. We would like to concentrate most of the growth of A_n on a set $\tilde{E}_n = \bigcup_{i=1}^{q_{n-1}} I_n^i$, where I_n^i is a subinterval of $(\,(i-1)/q_{n-1},\, i/q_{n-1}\,)$ of length d_n. More precisely, we choose A_n such that on each I_n^i it is linear with the slope $d_n^{-1} s_n (1 - 2^{-n})$.

Let E_n be the preimage of $\tilde{E}_n$ under h_{n-1}. Then $h_n(E_n) = A_n(\tilde{E}_n)$ and hence

$$(1_n) \quad \mu_n(E_n) > 1 - 2^{-n},$$

where μ_n is the measure with the distribution function h_n.

Now we will show how to choose a length d_n and two "scales" r_n and $\tilde{r}_n$, $n \ge 0$, such that

$$(2_n) \quad \frac{\log \Delta h_n(x, r_n)}{\log r_n} < \frac{1}{n} \quad \text{for any } x \in E_n,$$

$$(3_n) \quad \frac{\log \Delta h_n(x, \tilde{r}_n)}{\log \tilde{r}_n} > 1 - \frac{1}{n} \quad \text{for any } x \in [0, 1].$$

This means that for the measure μ_n the pointwise dimension "on the scale r_n" is less than $1/n$ on a set of μ_n-measure at least $1 - 2^{-n}$, and "on the scale $\tilde{r}_n$" it is at least $1 - 1/n$.

Let us introduce the following notations:

$$m_{n-1} = \min_{[0,1]} h'_{n-1} \quad \text{and} \quad M_{n-1} = \max_{[0,1]} h'_{n-1}$$

Note that the set E_n consists of q_{n-1} intervals whose lengths are bounded above by d_n/m_{n-1}. It follows that for $r_n = d_n/m_{n-1}$ and any $x \in E_n$,

$$\frac{\log \Delta h_n(x, r_n)}{\log r_n} \le \frac{\log (s_n(1 - 2^{-n}))}{\log d_n - \log m_{n-1}} \underset{d_n \to 0}{\longrightarrow} 0,$$

So we can take d_n so small that (2_n) holds. This completes the description of the choice of d_n and r_n and the construction of A_n.

Note that $\Delta h_n(x,r) \le 2rM_n$. This implies that

$$\frac{\log \Delta h_n(x,r)}{\log r} \ge 1 + \frac{\log(2M_n)}{\log r}.$$

Since $\log(2M_n)/\log r \to 0$ as $r \to 0$, there exists $\tilde r_n$ satisfying (3_n).

The distance between h_n and h is bounded by $\sum_{i=n}^{\infty} 1/q_i$. Since K_i, $i \ge n$, can be taken as large as we wish we may assume that h is so close to h_n in C^0 topology that the following properties hold true for the limit function h:

$(1'_n)$ $\mu(E_n) > 1 - 2^{-n+1}$,

$(2'_n)$ $\dfrac{\log \Delta h(x,r_n)}{\log r_n} < \dfrac{2}{n}$ for any $x \in E_n$,

$(3'_n)$ $\dfrac{\log \Delta h(x,\tilde r_n)}{\log \tilde r_n} > 1 - \dfrac{2}{n}$ for any $x \in [0,1]$.

Thus for any $n > 0$ there exists a set E_n and two "scales" r_n and $\tilde r_n$ satisfying $(1'_n) - (3'_n)$. Obviously r_n, $\tilde r_n \to 0$ as $n \to \infty$.

Take $x \in [0,1]$. If for any $N > 0$ there exist $n > N$ such that $x \in E_n$, then $\underline{d}_\mu(x) = 0$ and $\overline{d}_\mu(x) \ge 1$.

Otherwise, $x \in J = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty}([0,1] \setminus E_n)$. However $\mu(\bigcap_{n=m}^{\infty}([0,1] \setminus E_n)) = 0$ by $(2'_n)$. Hence $\mu(J) = 0$ and we conclude that for μ-almost all $x \in S^1$, $\underline{d}_\mu(x) = 0$ and $\overline{d}_\mu(x) \ge 1$.

It remains to note that $\overline{d}_\mu(x) \le 1$ μ-almost everywhere. This fact is probably well known and we include the following Lemma for the sake of completeness.

Lemma 2.1. *Let μ be a Borel probability measure on S^1. Then $\overline{d}_\mu(x) \le 1$ for μ-a.e. $x \in S^1$*

Proof. The function $\overline{d}_\mu(x)$ is measurable. If $\overline{d}_\mu(x) > 1$ on a set of positive measure then there exists $\delta > 0$ and a set $X \subset S^1$ of positive measure such that $\overline{d}_\mu(x) \ge 1 + 2\delta$ for all $x \in X$. It follows from the definition of the upper pointwise dimension that for any $\varepsilon > 0$ and any $x \in X$ there exists $r(x) \le \varepsilon$ such that $\mu(B(x, r(x))) \le r(x)^{1+\delta}$, where $B(x, r(x))$ is the interval in S^1 centered at x of length $2r(x)$. Since $X \subset \bigcup_{x\in X} B(x, \frac{1}{4}r(x)) \subset S^1$, by the Vitalie Covering Lemma there exists at most countable subset $\{x_n\}_{n\ge1}$ of X such that $X \subset \bigcup_n B(x_n, r(x_n))$ and the balls $B(x_n, \frac{1}{4}r(x_n))$ are disjoint. Then

$$\mu(X) \le \sum_n \mu(B(x_n, r(x_n))) \le \sum_n r(x_n)^{1+\delta} \le \varepsilon^\delta \sum_n r(x_n)$$

and hence

$$\sum_n \frac{1}{4} r(x_n) \ge \frac{\mu(X)}{4\varepsilon^\delta} > 1$$

for ε sufficiently small. This contradicts the fact that $B(x_n, \frac{1}{4}r(x_n))$ are disjoint intervals in S^1. $\square$

It follows that $\overline{d}_\mu(x) = 1$ for μ-a.e. $x \in S^1$. This completes the proof of Proposition 2.1. $\square$

We now construct a C^∞ circle diffeomorphism f with irrational rotation number such that for its unique invariant measure μ, the lower pointwise dimension is equal to a given number β, $0 < \beta < 1$, and the upper pointwise dimension is equal to 1 μ-almost everywhere.

Proposition 2.2. *Let $f_* : S^1 \to S^1$ be a C^∞ circle diffeomorphism such that $f_* = h_*^{-1} \circ R_{\tau_*} \circ h_*$, where h_* is a C^∞ circle diffeomorphism and R_{τ_*} is a circle rotation by τ_*.*

Given β, $0 < \beta < 1$, in any C^∞ neighborhood of f_* there exists a C^∞ diffeomorphism $f : S^1 \to S^1$ with irrational rotation number τ such that

1. *f is conjugate to the rotation R_τ;*
2. *the conjugacy map h is Hölder continuous with Hölder exponent β;*
3. *if μ is the invariant measure for f then $\underline{d}_\mu(x) = \beta$ and $\overline{d}_\mu(x) = 1$ for μ-a.e. $x \in S^1$.*

Proof. We follow the same approach as in the proof of Proposition 2.1 but we would like to make $\log \Delta h_n(x, r_n)/\log r_n$ close to β rather than to 0. For this we make the following modifications.

We choose the period s_n of the function a_n smaller than $1/q_{n-1}$ in the form $s_n = 1/l_n q_{n-1}$. Then we take $\tilde{E}_n = \bigcup_{i=1}^{l_n q_n} I_n^i$, where I_n^i is a subinterval of $(\,(i-1)/l_n q_{n-1},\, i/l_n q_{n-1}\,)$ of length d_n. We again concentrate most of the growth of A_n on $\tilde{E}_n$. We take A_n to be linear on each interval I_n^i with the slope $d_n^{-1} s_n(1 - 2^{-n})$. We may also assume that s_n/d_n is the upper bound for the derivative of A_n. Let us again introduce the following notations:

$$m_{n-1} = \min_{[0,1]} h'_{n-1} \quad \text{and} \quad M_{n-1} = \max_{[0,1]} h'_{n-1}$$

The preimage E_n of $\tilde{E}_n$ under h_{n-1} consists of $l_n q_{n-1}$ intervals whose lengths are bounded above by d_n/m_{n-1} and below by d_n/M_{n-1}. Then for any $x \in E_n$ and $r_n = d_n/m_{n-1}$ we have

$$\Delta h_n(x, r_n) \ge s_n(1 - 2^{-n}) \quad \text{and} \quad \Delta h_n(x, r_n) \le \frac{s_n}{d_n} \cdot M_{n-1} \cdot 2r_n = 2s_n \frac{M_{n-1}}{m_{n-1}},$$

where $\Delta h_n(x, r_n) = h_n(x + r) - h_n(x - r)$. Hence

$$\frac{\log s_n}{\log r_n} + \frac{\log\left(2M_{n-1}/m_{n-1}\right)}{\log r_n} \;\le\; \frac{\log \Delta h_n(x, r_n)}{\log r_n} \;\le\; \frac{\log s_n}{\log r_n} + \frac{\log\left(1 - 2^{-n}\right)}{\log r_n}.$$

We note that the error terms

$$\frac{\log(2M_{n-1}/m_{n-1})}{\log r_n} \quad \text{and} \quad \frac{\log(1-2^{-n})}{\log r_n}$$

are small once d_n is chosen so small that $r_n = d_n/m_{n-1}$ is small enough. Now we can choose d_n small and l_n large to satisfy the following properties

(1) The absolute values of the error terms are less then $\frac{1}{n}$;
(2) $(\log s_n)/(\log r_n) = \beta + \frac{1}{n}$;
(3) $r_n \le (M_{n-1}+1)^{-n}, \quad r_n \le (m_{n-1}/M_{n-1})^n, \quad s_n \le 2^{-(n+1)} s_{n-1}$,

where $s_n = 1/l_n q_{n-1}$.

The third property will be used to prove Hölder continuity of h.

So we conclude that for any $x \in E_n$

$$\beta \ < \ \frac{\log \Delta h_n(x, r_n)}{\log r_n} \ < \ \beta + \frac{2}{n}.$$

This means that for the measure μ_n the pointwise dimension "on the scale r_n" is about β on a set of large μ_n-measure.

Now it follows in the same way as in the previous proposition that $\underline{d}_\mu(x) \le \beta$ and $\overline{d}_\mu(x) = 1$ for μ-a.a. $x \in S^1$, where μ is the unique invariant measure for f.

It remains to show that $\underline{d}_\mu(x) \ge \beta$. Recall that $h = \lim h_n$ is the distribution function of μ.

Lemma 2.2. *h is Hölder continuous with the exponent β.*

Proof. It suffices to show that $|h_n(x)-h_n(y)| \le C|x-y|^\beta$ for all $x, y \in S^1$ and $n \ge 0$. We will prove by induction that for all $n \ge 0$ h_n has the following properties:

(*i*) $|h_n(x) - h_n(y)| \le |x-y|^\beta$ for all $x, y \in S^1$ with $|x-y| \le s_n$;

(*ii*) $|h_n(x) - h_n(y)| \le (4-2^{-n})|x-y|^\beta$ for all x, y with $|x-y| \ge s_n$.

For $h_0 = Id$ this holds true. We now show that h_n has properties (i) and (ii) provided that h_i with $i < n$ do. If $|x-y| \le r_n$ then

$$|h_n(x)-h_n(y)| \le \frac{s_n}{d_n} M_{n-1}|x-y| \le \frac{r_n^{\beta+\frac{1}{n}}}{r_n m_{n-1}} M_{n-1}|x-y| \le \le \frac{M_{n-1}}{m_{n-1}} |x-y|^{\beta+\frac{1}{n}} \le |x-y|^\beta$$

since $|x-y|^{\frac{1}{n}} \le r_n^{\frac{1}{n}} \le m_{n-1}/M_{n-1}$ by the choice of d_n. If $r_n \le |x-y| \le s_n$ then

$$|h_n(x) - h_n(y)| \le s_n(M_{n-1}+1) = r_n^{\beta+\frac{1}{n}}(M_{n-1}+1) \le |x-y|^\beta$$

since $r_n^{\frac{1}{n}} \le (M_{n-1}+1)$ again by the choice of d_n, and $|A_n(x) - A_n(y)| \le s_n$ if $|x-y| \le s_n$. So we conclude that h_n has property (i). If $s_n \le |x-y| \le s_{n-1}$ then

$$|h_n(x) - h_n(y)| \le 2s_n + |h_{n-1}(x) - h_{n-1}(y)| \le 2s_n + |x-y|^\beta \le 3|x-y|^\beta.$$

If $s_{n-1} \leq |x-y|$ then

$$|h_n(x)-h_n(y)| \leq 2s_n+|h_{n-1}(x)-h_{n-1}(y)| \leq 2s_n+(4-2^{-n+1})|x-y|^\beta \leq (4-2^n)|x-y|^\beta$$

since $2s_n \leq 2^{-n}s_{n-1}$ by the choice of l_n.

So we conclude that h_n has also property (ii). This completes the proof of the lemma. □

Lemma 2.2 implies that $\underline{d}_\mu(x) \geq \beta$ for all $x \in S^1$. This completes the proof of Proposition 2.2. □

Now we will prove Theorem 2.1 using the construction in Proposition 2.1.

Proof. Let $\tilde{f} \in D_I^r$. In any C^r-neighborhood of $\tilde{f}$ there exists a C^∞ diffeomorphism f_* with a Diophantine rotation number. Such diffeomorphisms are known to be C^∞-conjugate to corresponding rotations, i.e. $f_* = h_*^{-1}R_{\tau_*}h_*$, where h_* is a C^∞ circle diffeomorphism (see [5]). f_* can be used as a starting point for a sequence of iterations f_n constructed as in the proof of Proposition 2.1. Then the sequence f_n converges in C^r-topology to some diffeomorphism f which can be made as close to f_* as we wish and satisfies the following condition:

for any $n>0$ there exists a set E_n with $\mu(S^1 \setminus E_n) < 2^{-n+1}$ and positive numbers $r_n > \tilde{r}_n$ such that

$$\frac{\log \mu(B(x,r_n))}{\log r_n} < \frac{2}{n} \quad \text{for any } x \in E_n,$$
$$\frac{\log \mu(B(x,\tilde{r}_n))}{\log \tilde{r}_n} > 1-\frac{2}{n} \quad \text{for any } x \in S^1,$$

where μ is the unique invariant measure for f, and $r_n, \tilde{r}_n \leq \frac{1}{n}$. So we see that D_I^r contains a dense subset Z of diffeomorphisms satisfying the above condition.

For any diffeomorphism $f \in Z$ we can construct a sequence of its neighborhoods, $\{V_n^f\}_{n=1}^\infty$, such that any uniquely ergodic diffeomorphism g in V_n^f satisfies the condition

$$\frac{\log \nu(B(x,r_n))}{\log r_n} < \frac{3}{n} \quad \text{for any } x \in E_n,$$
$$\frac{\log \nu(B(x,\tilde{r}_n))}{\log \tilde{r}_n} > 1-\frac{3}{n} \quad \text{for any } x \in S^1,$$

where ν is the unique invariant measure for g, E_n, r_n and $\tilde{r}_n$ are the same as for f, and $\nu(S^1 \setminus E_n) < 2^{-n+2}$. Indeed, if f and g are sufficiently close in C^0-topology, their invariant measures are sufficiently close in the week topology.

Let $Y_0^r = \bigcap_{n=1}^\infty \bigcup_{f\in Z} V_n^f$. Then $Y_0^r \cap \overline{D_I^r}$ and $Y_0^r \cap D_I^r$ are residual subsets of $\overline{D_I^r}$ and D_I^r respectively.

Any uniquely ergodic diffeomorphism g in Y_0^r satisfies the above condition for some sequence of scales $\{r_n\}$ and $\{\tilde{r}_n\}$ which converge to 0. It follows that $\underline{d}_\nu(x) = 0$, $\overline{d}_\nu(x) = 1$ for ν-a.a. $x \in S^1$.

It is easy to see that the set E_n can be covered by $1/s_n$ balls of radius r_n (recall that s_n is the the period of the function a_n; see the proof of Proposition 2.1). Since $\log s_n / \log r_n \to 0$ as $n \to \infty$ we see that $\underline{\dim}_B(\bigcap_{n=k}^\infty E_n) = 0$ for any $k > 0$. Since $\nu(\bigcap_{n=k}^\infty E_n) > 1 - 2^{-n+3} \to 1$ we conclude that $\underline{\dim}_B \nu = 0$.

On the other hand, since $\nu(B(x, \tilde{r}_n)) < \tilde{r}_n^{1-\frac{3}{n}}$ for any $x \in S^1$ the minimal number N of balls of radius needed to cover a set of ν measure $1-\varepsilon$ is at least $(1-\varepsilon)\,\tilde{r}_n^{-(1-\frac{3}{n})}$. Hence

$$\frac{\log N}{-\log \tilde{r}_n} \geq 1 - \frac{3}{n} + \frac{\log(1-\varepsilon)}{-\log \tilde{r}_n} \underset{n\to\infty}{\longrightarrow} 1$$

and we conclude that $\overline{\dim}_B \nu \geq 1$. Since

$$0 \leq \dim_H \nu \leq \underline{\dim}_B \nu \leq \overline{\dim}_B \nu \leq 1$$

for any finite measure on S^1 we see that

$$\dim_H \nu = \underline{\dim}_B \nu = 0 \quad \text{and} \quad \overline{\dim}_B \nu = 1.$$

We conclude that any uniquely ergodic diffeomorphism g in Y_0^r lies in Y^r. Since the set of the diffeomorphisms in $\overline{D_I^r}$ which are not uniquely ergodic is of the first category, we conclude that Y^r is a residual subset in both D_I^r and $\overline{D_I^r}$.

□

The proof of Theorem 2.2 uses Proposition 2.2 and follows the corresponding steps of the proof of Theorem 2.1 almost identically. The lower bound for the Hausdorff dimension of the measure is provided in this case by the following fact (see [8]): if $\underline{d}_\mu(x) \geq \beta$ for μ-a.a. x then $\dim_H \mu \geq \beta$.

We now complete the section with the proof of Corollary 2.1.

Proof. It suffices to show that for any open and dense subset $U \subset D_I^r$ there exists a residual subset $T \subset [0,1]$ such that for any $\tau \in T$ the intersection $U \cap D_\tau^r$ is open and dense in D_τ^r. Since $U \cap D_\tau^r$ is open in the induced topology we only need to check whether it is dense.

Let us suppose that there exists a subset $S \subset [0,1]$ of the second category such that for any $\tau \in S$ the intersection $U \cap D_\tau^r$ is not dense in D_τ^r. In other words for any $\tau \in S$ there exist $f_\tau \in D_\tau^r$ and $r_\tau > 0$ such that $B(f_\tau, r_\tau) \cap D_\tau^r \cap U = \emptyset$, where $B(f_\tau, r_\tau)$ is the ball in D^r centered at f_τ of radius r_τ. Then for some $\epsilon > 0$ there exists $S_1 \subset S$ of the second category in $[0,1]$ such that $r_\tau > 3\epsilon$ for all $\tau \in S_1$. Since D_I^r is second countable there exists a countable ϵ-spanning set $\{g_n\} \subset D_I^r$. Then for some $i > 0$ there exists $S_2 \subset S_1$ of the second category in $[0,1]$ such that $f_\tau \in B(g_i, \epsilon)$ for all $\tau \in S_2$. Set $I = \tau(B(g_i, \epsilon))$ and by $I_u = \tau(B(g_i, \epsilon)) \cap U$, where $\tau : D^r \to [0,1]$ is the rotation number function. We obtain $S_2 \subset I$ and $S_2 \cap I_u = \emptyset$

since for all $\tau \in S_2$ we have $B(f_\tau, 3\epsilon) \cap D^r_\tau \cap U = \emptyset$ and $f_\tau \in B(g_i, \epsilon)$ whence $B(g_i, \epsilon) \cap D^r_\tau \cap U = \emptyset$. We note that I_u is open in $I \setminus \mathbb{Q}$ and $I \setminus I_u$ is of the second category in I since S_2 is. We conclude that $I \setminus I_u$ has nonempty interior. It follows that there exists an interval $I_s \subset I$ such that $I_s \cap I_u = \emptyset$. This implies that the set $\tau^{-1}(I_s) \cap B(g_i, \epsilon) \cap D^r_I$ is open in D^r_I and does not intersect U. This contradicts to the fact that U is dense and completes the proof of the corollary.

□

3. Circle Homeomorphisms

In the previous section we have shown that for a circle diffeomorphism we can make the lower pointwise dimension of its invariant measure μ equal to any number between 0 and 1. We do not know whether there exists a circle diffeomorphism such that $\overline{d}_\mu(x) = \gamma < 1$ for μ-a.e. $x \in S^1$. However we can obtain such pinching in the case of Hölder circle homeomorphisms. Moreover, we can construct Hölder circle homeomorphisms such that the pointwise dimension exists almost everywhere and is equal to a given number α, $0 < \alpha < 1$. We show that such homeomorphisms are dense in the set of all circle homeomorphisms with a given irrational rotation number.

Denote by H_τ, $\tau \in [0,1] \setminus \mathbb{Q}$, the set of all homeomorphisms of S^1 with rotation number τ.

Theorem 3.1.

(1) For any β, γ, $0 < \beta < \gamma \le 1$, the set of all Hölder homeomorphisms whose invariant measure has lower pointwise dimension equal to β and upper pointwise dimension equal to γ for a.e. $x \in S^1$ is everywhere dense in H_τ.

(2) For any $\alpha \in (0,1]$ the set of all Hölder homeomorphisms whose invariant measure has pointwise dimension α for a.e. $x \in S^1$ is everywhere dense in H_τ.

The proof of Theorem 3.1 is based on the following proposition.

Proposition 3.1.

(1) For any β, γ such that $0 < \beta < \gamma \le 1$ the set of all Borel probability measure μ on S^1 such that $\underline{d}_\mu(x) = \beta$ and $\overline{d}_\mu(x) = \gamma$ for μ-a.e. $x \in S^1$ is everywhere dense (in the week topology) in the set of all Borel probability measures on S^1.

(2) For any $\alpha \in (0,1]$ the set of all Borel probability measures ν on S^1 such that $d_\nu(x) = \alpha$ for ν-a.e. $x \in S^1$ is everywhere dense in the set of all Borel probability measures on S^1.

Proof. To obtain measures with desired properties on S^1 we first construct their counterparts on the symbolic space

$$\Omega_2 = \{\omega = (\omega_0 \omega_1 \dots) : \ \omega_i \in \{0,1\}, \ i \in \mathbb{N}_0\}.$$

Then we use the binary coding of the unit interval to carry the measures to S^1.

(1) Let us fix β and γ such that $0<\beta<\gamma\le 1$ and take the numbers $p,q,\tilde p,\tilde q$ such that

$$0<p\le q<1,\quad p+q=1,\quad p\log p+q\log q=\beta\log\frac{1}{2},$$
$$0<\tilde p\le \tilde q<1,\quad \tilde p+\tilde q=1,\quad \tilde p\log \tilde p+\tilde q\log \tilde q=\gamma\log\frac{1}{2}.$$

Let

$$s_n^0=p,\ \ s_n^1=q\quad\text{for }\ 2^{(2k)!}\le n<2^{(2k+1)!}$$
$$s_n^0=\tilde p,\ \ s_n^1=\tilde q\quad\text{for }\ 2^{(2k+1)!}\le n<2^{(2k+2)!}.$$

For any cylinder $C_{\omega_m\dots\omega_n}$ we set $\hat\mu\left(C_{\omega_m\dots\omega_n}\right)=\prod_{i=m}^n s_i^{\omega_i}$.

Consider the independent random variables

$$\xi_i=\begin{cases}\log p, & \text{if } \ \omega_i=0\ \text{ and }\ 2^{(2k)!}\le i<2^{(2k+1)!}\\ \log q, & \text{if } \ \omega_i=1\ \text{ and }\ 2^{(2k)!}\le i<2^{(2k+1)!}\\ \log \tilde p, & \text{if } \ \omega_i=0\ \text{ and }\ 2^{(2k+1)!}\le i<2^{(2k+2)!}\\ \log \tilde q, & \text{if } \ \omega_i=1\ \text{ and }\ 2^{(2k+1)!}\le i<2^{(2k+2)!}\end{cases}$$

Denote the expectation and the dispersion of ξ_i by A_i and D_i respectively. We have that

$$A_i=\int\limits_{\Omega_2}\xi_i\,d\mu=\begin{cases}\beta\log\frac12 & \text{if }\ 2^{(2k)!}\le i<2^{(2k+1)!}\\ \gamma\log\frac12 & \text{if }\ 2^{(2k+1)!}\le i<2^{(2k+2)!};\end{cases}$$

$$D_i=\int_{\Omega_2}|\xi_i-A_i|^2\,d\mu.$$

One can see that D_i is bounded by a constant independent of i. Therefore $\sum_{i=0}^\infty i^{-2}D_i<\infty$, and the Law of Large Numbers yields:

$$\lim_{n\to\infty}\left(\frac1n\sum_{i=0}^{n-1}\xi_i(\omega)-\frac1n\sum_{i=0}^{n-1}A_i\right)=0\quad\text{for }\ \hat\mu\text{-a.e. }\ \omega\in\Omega_2,$$

in particular,

$$\liminf_{n\to\infty}\left(\frac1n\sum_{i=0}^{n-1}\xi_i(\omega)\right)\ =\ \liminf_{n\to\infty}\left(\frac1n\sum_{i=0}^{n-1}A_i\right)\ =\ \beta\log\frac12,$$
$$\limsup_{n\to\infty}\left(\frac1n\sum_{i=0}^{n-1}\xi_i(\omega)\right)\ =\ \limsup_{n\to\infty}\left(\frac1n\sum_{i=0}^{n-1}A_i\right)\ =\ \gamma\log\frac12$$

for $\hat\mu$-a.e. $\omega\in\Omega_2$. It follows that for $\hat\mu$-a.e. $\omega\in\Omega_2$

$$\liminf_{n\to\infty}\frac{\frac1n\log\hat\mu(C_{\omega_0\dots\omega_{n-1}})}{\log\frac12}\ =\ \liminf_{n\to\infty}\frac{\frac1n\sum_{i=0}^{n-1}\xi_i(\omega)}{\log\frac12}\ =\ \beta,$$

$$\limsup_{n\to\infty} \frac{\frac{1}{n}\log\hat{\mu}(C_{\omega_0\dots\omega_{n-1}})}{\log\frac{1}{2}} = \gamma.$$

Let us consider the binary coding $\phi : \Omega_2 \to [0,1]$ of the interval $[0,1]$. Recall that each number has at most two binary expansions and any irrational number has exactly one.

Fix a measure κ_0 on S^1. Consider a measure κ with no atoms which is positive on open intervals and close to κ_0 in the week topology. Let $\hat{\kappa}$ be its pull back to Ω_2 by ϕ.

Fix $n \in \mathbb{N}$. For any cylinder $C_{\omega_0\dots\omega_m}$ set

$$\hat{\kappa}_j(C_{\omega_0\dots\omega_m}) = \begin{cases} \hat{\kappa}(C_{\omega_0\dots\omega_m}), & \text{if } m < j \\ \hat{\kappa}(C_{\omega_0\dots\omega_{n-1}}) \cdot \hat{\mu}(C_{\omega_n\dots\omega_m}), & \text{if } m \geq j \end{cases}$$

where $\hat{\mu}$ is the measure constructed above. It is easy to see that for any j we have that

$$\liminf_{n\to\infty} \frac{\frac{1}{n}\log\hat{\kappa}_j(C_{\omega_0\dots\omega_{n-1}})}{\log\frac{1}{2}} = \beta, \qquad \limsup_{n\to\infty} \frac{\frac{1}{n}\log\hat{\kappa}_j(C_{\omega_0\dots\omega_{n-1}})}{\log\frac{1}{2}} = \gamma$$

for $\hat{\kappa}_j$-a.e. $\omega \in \Omega_2$.

Let us denote by κ_j the push forward of $\hat{\kappa}_j$ to $[0,1]$ by ϕ. Clearly, the measure κ_j is close to κ for large j, positive on open intervals and has no atoms. To complete the proof of the second statement of the proposition it remains to prove the following lemma.

Lemma 3.1. *$\underline{d}_{\kappa_j}(x) = \beta$ and $\overline{d}_{\kappa_j} = \gamma$ for κ_j-a.e. $x \in S^1$.*

Proof. Note that $\phi(C_{\omega_0\dots\omega_{n-1}})$ is one of 2^n closed binary intervals of length 2^{-n}. So we see that $\phi^{-1}(B(x,2^{-n})) \supset C_{\omega_0\dots\omega_{n+1}}$ for any $x \in S^1$, where $\phi(\omega_0\omega_1\dots) = x$. It follows that, for κ_j-a.e. $x \in S^1$

$$\begin{aligned}
\underline{d}_{\kappa_j}(x) &= \liminf_{n\to\infty} \frac{\log\kappa_j(B(x,2^{-n}))}{\log 2^{-n}} \leq \lim_{n\to\infty} \frac{\log\hat{\kappa}_j(C_{\omega_0\dots\omega_{n+1}})}{n\,\log\frac{1}{2}} \\
&= \lim_{n\to\infty} \left(\frac{n+2}{n} \cdot \frac{\log\hat{\kappa}_j(C_{\omega_0\dots\omega_{n+1}})}{(n+2)\,\log\frac{1}{2}} \right) = \beta, \\
\overline{d}_{\kappa_j}(x) &= \limsup_{n\to\infty} \frac{\log\kappa_j(B(x,2^{-n}))}{\log 2^{-n}} \leq \lim_{n\to\infty} \frac{\log\hat{\kappa}_j(C_{\omega_0\dots\omega_{n+1}})}{n\,\log\frac{1}{2}} = \gamma.
\end{aligned}$$

To obtain the below estimates we introduce the following sets

$$B_k = \bigcup_{i=1}^{2^k} \left[\frac{i}{2^k} - \frac{1}{2^{k+[\sqrt{k}]}}, \frac{i}{2^k} + \frac{1}{2^{k+[\sqrt{k}]}} \right] \subset S^1 \quad \text{and} \quad G_m = S^1 \setminus (\bigcup_{k=m}^{\infty} B_k).$$

It is easy to see that for any $x \in G_m$ and any $n > m$, we have

$$\phi^{-1}(B(x, 2^{-(n+[\sqrt{n}])})) \subset C_{\omega_0 \dots \omega_{n-1}}.$$

Hence, for κ_j-a.e. $x \in G_m$,

$$\begin{aligned}
\underline{d}_{\kappa_j}(x) &= \liminf_{n\to\infty} \frac{\log \kappa_j(B(x, 2^{-(n+[\sqrt{n}])}))}{\log 2^{-(n+[\sqrt{n}])}} \ \geq \lim_{n\to\infty} \frac{\log \hat{\kappa}_j(C_{\omega_0\dots\omega_{n-1}})}{(n+[\sqrt{n}]) \log \frac{1}{2}} \\
&= \lim_{n\to\infty} \left(\frac{n}{n+[\sqrt{n}]} \cdot \frac{\log \hat{\kappa}_j(C_{\omega_0\dots\omega_{n-1}})}{n \log \frac{1}{2}} \right) = \beta, \\
\overline{d}_{\kappa_j}(x) &= \limsup_{n\to\infty} \frac{\log \kappa_j(B(x, 2^{-(n+[\sqrt{n}])}))}{\log 2^{-(n+[\sqrt{n}])}} \ \geq \lim_{n\to\infty} \frac{\log \hat{\kappa}_j(C_{\omega_0\dots\omega_{n-1}})}{(n+[\sqrt{n}]) \log \frac{1}{2}} = \gamma.
\end{aligned}$$

For any $k > j$ we observe that $\kappa_j(B_k) \leq 2q^{\sqrt{n}}$, where $q < 1$ is from the construction of the measure $\hat{\nu}$. Hence, $\kappa_j(G_m) \nearrow 1$ as $m \to \infty$. It follows that $\underline{d}_{\kappa_j} \geq \beta$ and $\overline{d}_{\kappa_j} \geq \gamma$ for κ_j-a.e. $x \in S^1$, and this completes the proof of the lemma. □

This completes the proof of the first statement.

(2) Let us fix $\alpha \in (0, 1]$ and take the numbers p and q such that

$$0 < p \leq q < 1, \ \ p + q = 1 \ \text{ and } \ p \log p + q \log q = \alpha \log \frac{1}{2}.$$

Let us consider the Bernoulli measure $\nu = \nu(p, q)$ on Ω_2 which is defined as follows: for any cylinder

$$C_{\omega_m\dots\omega_n} = \{\, \omega' \in \Omega_2 : \ \omega'_i = \omega_i, \ m \leq i \leq n \,\},$$

$\hat{\nu}\,(C_{\omega_m\dots\omega_n}) = \prod_{i=m}^{n} s_i^{\omega_i}$, where $s_i^0 = p$ and $s_i^1 = q$.

This measure is ergodic with respect to the shift σ. Clearly, it has no atoms and is positive on any cylinder.

Consider the function

$$g(\omega) = \begin{cases} \log p, \ \text{if } \omega_0 = 0 \\ \log q, \ \text{if } \omega_0 = 1 \end{cases}$$

By the Birkhoff ergodic theorem

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g(\sigma^i(\omega)) = \int_{\Omega_2} g \, d\hat{\nu} = p \log p + q \log q \qquad \text{for } \hat{\nu}\text{-a.e. } \omega \in \Omega_2.$$

This implies that for $\hat{\nu}$-a.e. $\omega \in \Omega_2$,

$$\lim_{n\to\infty} \frac{\frac{1}{n} \log \hat{\nu}(C_{\omega_0\dots\omega_{n-1}})}{\log \frac{1}{2}} \ = \ \frac{p \log p + q \log q}{\log \frac{1}{2}} \ = \alpha.$$

We now use the measure $\hat{\nu}$ in to modify a given measure κ_0 in the same way as we used $\hat{\mu}$ in the proof of the first statement, and the rest of the proof follows similarly. □

Now we will prove Theorem 3.1.

Proof. Fix an irrational rotation number τ and consider a diffeomorphism $\tilde{f} \in H_\tau$. In any neighborhood of $\tilde{f}$ there exists a C^2 circle diffeomorphism f_* with an irrational rotation number. By the Denjoy Theorem it is conjugate to the corresponding rotation: $f_* = h_*^{-1} \circ R_{\tau_*} \circ h_*$. Consider the homeomorphism $f_0 = h_*^{-1} \circ R_\tau \circ h_*$. It is close to $\tilde{f}$ and has the same rotation number.

(2) Let κ_0 be the invariant measure for f_0. Consider a sequence of measures κ_j without atoms and positive on open intervals which have pointwise dimension equal to α for κ_j-a.a. $x \in S^1$ and weekly converge to κ (constructed as in Proposition 3.1). Let h_n be the distribution function of κ_j and $f_n = h_n^{-1} \circ R_\tau \circ h_n$. Then it is easy to see that f_n converge uniformly to f_0 and f_n^{-1} converge uniformly to f_0^{-1}.

Lemma 3.2. *The homeomorphisms f_n constructed above are Hölder continuous with Hölder exponent* $\log q/2\log p$.

Proof. Let A and B be binary intervals, $|A| = 2^{-m}$, $|B| = 2^{-k}$, such that $\kappa_j(A) \le \kappa_j(B)$. We will show that $m/k \ge \log q/2\log p$ i.e. $|A| \le |B|^{\frac{\log q}{2\log p}}$.

Recall that $\phi^{-1}(A) = C_{\omega_0\dots\omega_{m-1}}$ and $\phi^{-1}(B) = C_{\omega'_0\dots\omega'_{k-1}}$ for some $(\omega_0 \dots \omega_{m-1})$ and $(\omega'_0 \dots \omega'_{k-1})$ (up to countably many elements). We can assume that $m, k > n$. Then

$$\hat{\kappa}_j(C_{\omega_0\dots\omega_m}) = \hat{\kappa}(C_{\omega_0\dots\omega_{n-1}}) \prod_{i=n}^{m-1} s_i^{\omega_i} \le \hat{\kappa}(C_{\omega'_0\dots\omega'_{n-1}}) \prod_{j=n}^{k-1} s_j^{\omega'_j} = \hat{\kappa}_j(C_{\omega'_0\dots\omega'_m}).$$

Let

$$M_n = \max \frac{\hat{\kappa}(C_{\omega_0\dots\omega_n})}{\hat{\kappa}(C_{\omega'_0\dots\omega'_n})},$$

where maximum is taken over all cylinders of length $n+1$. The ratio m/k is the smallest when $s_i^{\omega_i} = p$, $i = n, \dots, m-1$ and $s_j^{\omega'_j} = q$, $j = n, \dots k-1$. Therefore $p^{m-n} \le Mq^{k-n}$ and

$$\frac{m}{k} \ \ge\ \frac{\log q}{\log p} + \frac{\log M + n\log(p/q)}{k\log p} \ \ge\ \frac{\log q}{2\log p}$$

if k is big enough.

Let I be an interval, $A \subset I$ be a binary interval (i.e. the image of a cylinder in Ω_2 under ϕ) of the largest possible length and $B \supset f_n(I)$ a binary interval of the smallest possible length. Then $\kappa_j(A) = \kappa_j(f_n(A)) \le \kappa_j(f_n(I)) \le \kappa_j(B)$. Hence

$$|I| \le 2|A| \le 2|B|^{\frac{\log q}{2\log p}} \le 2(2|f_n(I)|)^{\frac{\log q}{2\log p}} = 2^{\frac{\log q}{2\log p}+1}|f_n(I)|^{\frac{\log q}{2\log p}}.$$

The same argument shows that $|f_n(I)| \le 2^{\frac{\log q}{2\log p}+1}|I|^{\frac{\log q}{2\log p}}$. □

This completes the proof of the second part of the theorem. The first part can be proven similarly. □

4. Analytic Circle Diffeomorphisms

Let us fix an annulus $A \subset \mathbb{C}$ containing S^1 and denote by $D^\omega = D^\omega(A) \subset \mathrm{Diff}^\omega(S^1)$ the set of all orientation-preserving circle diffeomorphisms f such that f and f^{-1} extend to analytic functions on A. We endow D^ω with the topology of uniform convergence on compact subsets of A. Denote by D^ω_I the subset of D^ω consisting of all diffeomorphisms with irrational rotation number.

Let Y^ω be the subset of D^ω_I consisting of diffeomorphisms f such that

(1) $\underline{d}_\mu(x) = 0$ and $\overline{d}_\mu(x) = 1$ for μ-a.a. $x \in S^1$,

(2) $\dim_H \mu = \underline{\dim}_B \mu = 0$ and $\overline{\dim}_B \mu = 1$,

where μ is the invariant measure for f. The following statements are analytic counterparts of Theorem 2.1 and Corollary 2.1.

Theorem 4.1. *Y^ω is a residual subset of both D^ω_I and $\overline{D^\omega_I}$.*

The proof of Theorem 4.1 is based on Propositions 4.2 and 4.3 below.

Let D^ω_τ be the set of all diffeomorphisms in D^ω with rotation number τ.

Corollary 4.1. *There exists a set $T^\omega \subset [0,1] \setminus \mathbb{Q}$ which is a residual subset of $[0,1]$ such that for any $\tau \in T^\omega$, $Y^\omega \cap D^\omega_\tau$ is a residual subset of D^ω_τ.*

Proof. The proof is identical to the proof of Corollary 2.1. □

Remark 4.1. *The set T^ω has zero Lebesgue measure and zero Hausdorff dimension (compare to Remark 2.1).*

We have a natural projection $\pi : \mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z}$. This provides a lift of a diffeomorphism $f : S^1 \to S^1$ to a diffeomorphism $F : \mathbb{R} \to \mathbb{R}$ such that $f \circ \pi = \pi \circ F$.

Let f be an analytic orientation-preserving diffeomorphism. We say that f satisfies the property $(\star)$ if for any $\alpha \in [0,1]$, no power of the diffeomorphism

$$f_\alpha = f + \alpha \pmod 1$$

is the identity map.

The following proposition proves the existence of diffeomorphisms satisfying the property $(\star)$.

Proposition 4.1. *Let $f : S^1 \to S^1$ be an orientation-preserving diffeomorphism such that it is not a rotation and its lift $F : \mathbb{R} \to \mathbb{R}$ extends to an entire function. Then f satisfies the property $(\star)$.*

Proof. Suppose $f_\alpha^q = Id$ for some $q \in \mathbb{N}$ and $\alpha \in [0,1]$. Then $F_\alpha^q = Id + p$ on $\mathbb{C}$ for some $p \in \mathbb{Z}$. This implies that $F : \mathbb{C} \to \mathbb{C}$ is a bijection. Since F is entire it must be a liner function. It follows that the diffeomorphism f is a rotation. □

The following proposition shows that property $(\star)$ diffeomorphisms are typical.

Proposition 4.2. *The diffeomorphisms in D_I^ω satisfying the property $(\star)$ form a residual subset of $\overline{D_I^\omega}$.*

The following proof was given by Keith Burns.

Proof. We will prove that the set

$$\{ f \in D_I^\omega : \ f_\alpha^n \neq Id \ \text{ for all } \ \alpha \in [0,1] \ \text{ and } \ n \geq 1 \}$$

is a residual subset of $\overline{D_I^\omega}$. It suffices to show that for every $n \geq 1$ the set

$$G_n = \{ f \in D_I^\omega : \ f_\alpha^n \neq Id \ \text{ for all } \ \alpha \in [0,1] \}$$

is open and dense in $\overline{D_I^\omega}$.

Fix $n \geq 1$. It is easy to see that the complement of G_n is closed so it is enough to check that G_n is dense in $\overline{D_I^\omega}$. Let $U \subset \overline{D_I^\omega}$ be an open set. We will show that $U \cap G_n \neq \emptyset$. Let us take a diffeomorphism $f \in U$ with irrational rotation number and suppose that $f \notin G_n$.

Let F be a lift of f, then $F_\alpha = F + \alpha$ is a lift of f_α. The equality $f_\alpha^n = Id$ is equivalent to $F_\alpha^n = Id + p$ for some $p \in \mathbb{Z}$. Since $F_\alpha^n(x)$ is increasing in α there are only finitely many values of α in $[0,1]$ for which $f_\alpha^n = Id$. Let us denote these values by $\alpha_1, \dots, \alpha_k$.

Let $E = [0,1] \setminus (I_1 \cup \cdots \cup I_k)$, where I_j, $j = 1, \dots, k$, are open intervals centered at α_i of length $(\max(2, \sup|f'|))^{-(n+1)}$. Since $f_\alpha^n \neq Id$ for any $\alpha \in E$ there exists a neighborhood $U_0 \subset U$ of f such that $g_\alpha^n \neq Id$ for any $g \in U_0$ and any $\alpha \in E$.

Lemma 4.1. *Let $f \in D_I^\omega$ be such that $F_\alpha = Id + p$ for some $p \in \mathbb{Z}$ and $\alpha \in [0,1]$. Then in any neighborhood of f there exists $\tilde{f} \in D_I^\omega$ such that for its lift $\tilde{F}$,*

$$\tilde{F}_\alpha^n(x') < x' + p \quad \textit{and} \quad \tilde{F}_\alpha^n(x'') > x'' + p$$

for some $x', x'' \in \mathbb{R}$.

Proof. Let us fix $x \in S^1$ and consider its orbit $x, fx, ..., f^m x$, where $m+1$ is the minimal period of x. There exists an analytic flow ϕ^t on S^1 for which $x, fx, ..., f^m x$ are repelling fixed points. In other words, for $t > 0$, $1 \leq i \leq m$, and for all y sufficiently close to $f^i x$ we have $\Phi^t y > y$ if $y > x$ and $\Phi^t y < y$ if $y < x$, where Φ^t is the lift of ϕ^t such that $\Phi^0 = Id$. It is easy to see that if $x' < x$ and $x'' > x$ are sufficiently close to x and if $t > 0$ is small then $\tilde{f} = f \circ \phi^t$ satisfies the conditions of the lemma. We note that since f has irrational rotation number, $\tilde{f}$ can be also chosen to have an irrational rotation number and can be made as close to f as we wish. □

Since $F_{\alpha_1} = Id + p_1$ for some $p_1 \in \mathbb{Z}$, Lemma 4.1 implies that there exists $\tilde{f} \in U_0$ such that $\tilde{F}^n_{\alpha_1}(x') < x' + p_1$ and $\tilde{F}^n_{\alpha_1}(x'') > x'' + p_1$ for some $x', x'' \in \mathbb{R}$. It follows that for $\alpha \in I_1$ we have $\tilde{F}^n_{\alpha}(x') < x' + p_1$ if $\alpha < \alpha_1$, and $\tilde{F}^n_{\alpha}(x'') > x'' + p_1$ if $\alpha > \alpha_1$.

If $\tilde{f}$ is chosen close to f then $|\tilde{F}_\alpha(x) - F_{\alpha_1}(x)| < |I_1|$ for any $\alpha \in I_1$ and $x \in \mathbb{R}$. So by the choice of the length $|I_1|$ it follows that

$$|\tilde{F}^n_\alpha(x) - (x + p_1)| = |\tilde{F}^n_\alpha(x) - F^n_{\alpha_1}(x)| < 1$$

and therefore

$$x + p_1 - 1 < \tilde{F}^n_\alpha(x) < x + p_1 + 1$$

for any $\alpha \in I_1$ and $x \in \mathbb{R}$. So we conclude that $\tilde{f}_\alpha \neq Id$ for $\alpha \in I_1 \cup E$.

We can choose a neighborhood $U_1 \subset U_0$ of $\tilde{f}$ such that $g^n_\alpha \neq Id$ for any $g \in U_1$ and any $\alpha \in E \cup I_1$. The proposition now follows by consecutive application of Lemma 4.1. ☐

Proposition 4.3. *Let $f \in D^\omega_I$ satisfy the property ($\star$). Then for any $\delta > 0$ there exists α_*, $0 < \alpha_* < \delta$, such that the diffeomorphism $f_{\alpha_*} = f + \alpha_* \,(mod\, 1)$ has irrational rotation number and for its unique invariant measure μ, $\underline{d}_\mu(x) = 0$ and $\overline{d}_\mu(x) = 1$ for μ-a.a. $x \in S^1$.*

Proof. Our construction uses the method of constructing analytic circle diffeomorphisms with singular conjugacy described in [6].

Consider the family of analytic circle diffeomorphisms

$$f_\alpha = f + \alpha \,(\text{mod}\, 1), \quad \text{where } \ 0 < \alpha < \delta.$$

Denote by $\tau(\alpha)$ the rotation number of f_α. This family has the following properties:

(a) $\tau(\alpha)$ is nondecreasing in α;

(b) f_α never has infinitely many periodic points (by the property ($\star$));

(c) Suppose that α is the right endpoint of some interval J such that $\tau(\alpha') = p/q$, $\alpha' \in J$. Then the lift of f_α satisfies $F^q_\alpha - Id - p \geq 0$ and the zeros of $F^q_\alpha - Id - p$ project to the periodic orbits of f_α. Hence all periodic orbits of f_α are semistable, i.e. attract on one side and repel on the other side. All non-periodic points move in the same direction under iterations of f^q_α (see [6] for more details).

We will inductively choose two sequences of numbers $\{\alpha_n\}_{n=1}^\infty$ and $\{\tilde{\alpha}_n\}_{n=1}^\infty$ satisfying:

(1) α_n, $\tilde{\alpha}_n < \delta/2$;
(2) $\alpha_{n-1} < \tilde{\alpha}_{n-1} < \alpha_n$;
(3) $\alpha_n = \max \tau^{-1}(p_n/q_n)$; $\tau(\tilde{\alpha}_n)$ is Diophantine;
(4) $p_n/q_n - p_{n-1}/q_{n-1} < \left(2(n-1)^2 \max\limits_{1\leq k\leq n-1} q_k^2\right)^{-1}$.

Let P_n be the number of periodic points of f_{α_n}. Choose $r_n > 0$ so small that $r_n \le (2^n P_n)^{-n}$ and open intervals of length r_n centered at the periodic points of f_{α_n} are disjoint. Let I_n be the union of these intervals. By (c), there exists a number $k(\alpha_n, I_n)$ such that the orbit of any point $x \in S^1$ has at most $k(\alpha_n, I_n)$ points outside I.

Any sufficiently small perturbation of f_{α_n} satisfies the following two properties.

(i) The time of the first return to I_n is bounded above by $k(\alpha_n, I_n) + 1$.

(ii) Once a trajectory enters I_n next $2^{n+2} k(\alpha_n, I_n)$ iterations belong to I_n.

It follows that there exists $\varepsilon_n > 0$ such that for any $\alpha \in [\alpha_n, \alpha_n + \varepsilon_n)$

$$\frac{1}{N} \sum_{k=0}^{N} \chi_{I_n}(f_\alpha^k(x)) \; > \; 1 - \frac{1}{2^n}$$

for any $N > 2^{n+2} k(\alpha_n, I_n)$ and any $x \in S^1$.

We will choose α_m, $m > n$, and $\tilde{\alpha}_l$, $l \ge n$, such that

(5) $\alpha_m, \tilde{\alpha}_l \in [\alpha_n, \alpha_n + \varepsilon_n/2\,)$.

Now we choose a number $\tilde{\alpha}_n \in [\alpha_n, \alpha_n + \varepsilon_n/2\,)$ such that $\tau\,(\tilde{\alpha}_n)$ is Diophantine. This implies that $f_{\tilde{\alpha}_n}$ is smoothly conjugate to the rotation $R_{\tau\,(\tilde{\alpha}_n)}$ ([5]). Denote by $\mu_{\tilde{\alpha}_n}$ the unique invariant measure corresponding to $f_{\tilde{\alpha}_n}$. Since $\mu_{\tilde{\alpha}_n}$ has a smooth density there exists $\tilde{r}_n > 0$ such that for any $x \in S^1$

$$\frac{\log \mu_{\tilde{\alpha}_n}(B(x, \tilde{r}_n))}{\log \tilde{r}_n} > 1 - \frac{1}{2^{n+1}}.$$

There exists $\tilde{\varepsilon}_n > 0$ such that for any $\alpha \in [\tilde{\alpha}_n, \tilde{\alpha}_n + \tilde{\varepsilon}_n)$ with irrational $\tau\,(\alpha)$

$$\frac{\log \mu_\alpha(B(x, \tilde{r}_n))}{\log \tilde{r}_n} > 1 - \frac{1}{2^n}$$

for all $x \in S^1$.

We will choose α_m, $\tilde{\alpha}_m$, $m > n$, such that

(6) $\alpha_m,\ \tilde{\alpha}_m \in [\tilde{\alpha}_n, \tilde{\alpha}_n + \tilde{\varepsilon}_n/2\,)$.

Let $\alpha_* = \lim\limits_{n\to\infty} \alpha_n$. Note that the limit exists since the sequence $\{\alpha_n\}$ is monotone and bounded from above, and $\alpha_* \le \delta/2$.

Now we will show that $\tau(\alpha_*)$ is indeed irrational. By continuity, $\tau(\alpha_*) = \lim_{n\to\infty} p_n/q_n$. Therefore,

$$\tau(\alpha_*) - p_n/q_n = \sum_{k=n}^{\infty} (\, p_{k+1}/q_{k+1} - p_k/q_k\,) \le \sum_{k=n}^{\infty} \frac{1}{2k^2 \max\limits_{1\le i\le k} q_i^2} \le \sum_{k=n}^{\infty} \frac{1}{2k^2 q_n^2} \le \frac{\pi^2}{12 q_n^2} < \frac{1}{q_n^2}.$$

On the other hand, if $\tau(\alpha_*) = p/q$ then for $n \in \mathbb{N}$ such that $q_n > q$ we have

$$p/q - p_n/q_n \; = \; \frac{pq_n - qp_n}{qq_n} \; > \; \frac{1}{q_n^2}.$$

Let us denote by μ the invariant measure for f_{α_*}. Since $\alpha_* \in (\alpha_n, \alpha_n + \varepsilon_n/2\,]$,

$$\frac{1}{N}\sum_{k=0}^{N} \chi_{I_n}(f^k_{\alpha_*}(x)) \;>\; 1 - 2^{-n}$$

for all $N > 2^{n+2}k(\alpha_n, I_n)$, and hence $\mu(I_n) \geq 1 - 2^{-n}$.

Recall that the set I_n consists of P_n open intervals of length r_n. Let $\hat{I}_n$ be the union of those intervals $\Delta_n \subset I_n$ for which $\mu(\Delta_n) \geq (2^n P_n)^{-1}$. Since $r_n \leq (2^n P_n)^{-n}$ we have

$$\frac{\log \mu(B(x, r_n))}{\log r_n} \;\leq\; \frac{1}{n}$$

for any $x \in \hat{I}_n$. Therefore, if a point x belongs to $\hat{I}_n$ for infinitely many n then $\underline{d}_\mu(x) = 0$. Otherwise $x \in \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} (S^1 \setminus \hat{I}_n) = F$.

Note that $\mu(\hat{I}_n) \;\geq\; \mu(I_n) - P_n(2^n P_n)^{-1} \;\geq\; 1 - 2^{-n+1}$. It follows that $\mu(\bigcap_{n=m}^{\infty}(S^1 \setminus \hat{I}_n)) = 0$, and $\mu(F) = 0$. So we conclude that $\underline{d}_\mu(x) = 0$ for μ-a.a. $x \in S^1$.

Since $\alpha_* \in (\tilde{\alpha}_n, \tilde{\alpha}_n + \tilde{\varepsilon}/2\,]$ for any n, $\overline{d}_\mu(x) \geq 1$ for all $x \in S^1$. This together with Lemma 2.1 implies that $\overline{d}_\mu(x) = 1$ for μ-a.a. $x \in S^1$ □

Now we will prove Theorem 4.1 using Propositions 4.2 and 4.3. The proof is similar to the proof of Theorem 2.1.

Proof. By Proposition 4.2 uniquely ergodic property ($\star$) diffeomorphisms are dense in $\overline{D^\omega_I}$. Hence using Proposition 4.3 we can construct a dense subset $Z \subset \overline{D^\omega_I}$ with the following property. For every $f \in Z$ and $n > 0$ there exist positive numbers $\tilde{r}_n < r_n < 2^{-n}$ and a set $\hat{I}_n$ with $\mu(\hat{I}_n) > 1 - 2^{-n+1}$ (constructed in the proof of Proposition 4.3) such that

$$\frac{\log \mu(B(x, r_n))}{\log r_n} < \frac{1}{n} \quad \text{for any } x \in \hat{I}_n,$$
$$\frac{\log \mu(B(x, \tilde{r}_n))}{\log \tilde{r}_n} > 1 - \frac{1}{n} \quad \text{for any } x \in S^1,$$

where μ is the unique invariant measure for f.

For any diffeomorphism $f \in Z$ we can construct a sequence of its neighborhoods, $\{V^f_n\}_{n=1}^{\infty}$, such that for any uniquely ergodic diffeomorphism g in V^f_n and its the unique invariant measure ν we have

$$\frac{\log \nu(B(x, r_n))}{\log r_n} < \frac{2}{n} \quad \text{for any } x \in \hat{I}_n,$$
$$\frac{\log \nu(B(x, \tilde{r}_n))}{\log \tilde{r}_n} > 1 - \frac{2}{n} \quad \text{for any } x \in S^1,$$

and $\nu(\hat{I}_n) > 1 - 2^{-n+2}$, where $\hat{I}_n$, r_n and $\tilde{r}_n$ are the same as for f. Indeed, if f and g are sufficiently close in C^0-topology, their invariant measures are sufficiently close in the week topology.

Let $Y_0^\omega = \bigcap_{n=1}^\infty \bigcup_{f\in Z} V_n^f$. Then $Y_0^\omega \cap \overline{D_I^\omega}$ and $Y_0^\omega \cap D_I^\omega$ are residual subsets of $\overline{D_I^\omega}$ and D_I^ω respectively.

Any uniquely ergodic diffeomorphism g in Y_0^ω satisfies the above condition for some sequence of scales $\{r_n\}$ and $\{\tilde{r}_n\}$ which converge to 0. It follows that $\underline{d}_\nu(x) = 0$, $\overline{d}_\nu(x) = 1$ for ν-a.a. $x \in S^1$.

Since r_n, $\tilde{r}_n$ and $\hat{I}_n$ are as in the proof of Proposition 4.3, we see that the set $\hat{I}_n$ can be covered by at most P_n intervals of length $r_n \le (2^n P_n)^n$. Hence $\log P_n / \log r_n \to 0$ as $n \to \infty$ and we conclude that $\underline{\dim}_B(\bigcap_{n=k}^\infty \hat{I}_n) = 0$ for any $k > 0$. Since $\nu(\bigcap_{n=k}^\infty \hat{I}_n) > 1 - 2^{-n+3} \to 1$, it follows that $\underline{\dim}_B \nu = 0$.

On the other hand, since $\nu(B(x,\tilde{r}_n)) < \tilde{r}_n^{1-\frac{2}{n}}$ for any $x \in [0,1]$, the minimal number N of balls of radius needed to cover a set of ν-measure $1-\varepsilon$ is at least $(1-\varepsilon)\,\tilde{r}_n^{-(1-\frac{2}{n})}$. Hence

$$\frac{\log N}{-\log \tilde{r}_n} \ \ge\ 1 - \frac{2}{n} + \frac{\log(1-\varepsilon)}{-\log \tilde{r}_n} \underset{n\to\infty}{\longrightarrow} 1$$

and we conclude that $\overline{\dim}_B \nu \ge 1$. Since

$$0 \le \dim_H \nu \le \underline{\dim}_B \nu \le \overline{\dim}_B \nu \le 1$$

for any finite measure on S^1, we see that

$$\dim_H \nu = \underline{\dim}_B \nu = 0 \quad \text{and} \quad \overline{\dim}_B \nu = 1.$$

This implies that any uniquely ergodic diffeomorphism g in Y_0^ω lies in Y^ω. Since the set of the diffeomorphisms in $\overline{D_I^\omega}$ which are not uniquely ergodic is of the first category, we conclude that Y^ω is a residual subset in both D_I^ω and $\overline{D_I^\omega}$.

□

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