ON THE REGULARITY OF INTEGRABLE CONFORMAL STRUCTURES INVARIANT UNDER ANOSOV SYSTEMS

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Abstract. We consider conformal structures invariant under a volume-preserving Anosov system. We show that if such a structure is in L^p for suffiently large p, then it is continuous.

1. Introduction

In this paper we consider integrable conformal structures which are invariant under a volume preserving Anosov system. Our goal is to show that such structures are actually continuous. Our main results for Anosov diffeomorphisms and flows are stated in Section 2. In Section 3 we indicate some generalizations of these results, in particular, an extension to partially hyperbolic accessible diffeomorphisms.

This research was motivated by recent developments in the study of rigidity properties of conformal Anosov systems. The paper [dlL02] showed that conformal Anosov systems are locally differentiably rigid. These results were extended [dlLb]. More results on local and global differentiable rigidity of such systems were obtained in [Sad] and [KS03].

The above papers assumed that the conformal structures were continuous or bounded. Various arguments were developed there to show that continuous invariant structures are in fact differentiable. Nevertheless, many of the tools from the theory of quasi-conformal structures – e.g. the measurable Riemann mapping theorem – are designed to produce integrable conformal structures.

The goal of this paper is to bridge the gap between the integrable theory and the continuous/differentiable one. We will show that a conformal structure is continuous provided that it belongs to a certain L^p with p high enough. Once the conformal structure is known to be continuous, one can use the results of [Sad], [KS03] to obtain further regularity for the conformal structure and differential rigidity for the system. Bootstrap or regularity for measurable equations has applications in ergodic theory. See for example, [PP97, NP99].

The method of proof is based on the fact that the invariant structures satisfy a functional equation very similar to the equation for solutions of cohomology equations. Hence, some of the techniques of [dlLa] apply. Moreover, taking advantage of the form of the problem, the results for conformal structures can be made somewhat sharper than those for general solutions of cohomology equations.

2. Statement of results

Let $\mathcal{M}$ be a m-dimensional C^∞ compact manifold, and $f : \mathcal{M} \to \mathcal{M}$ be a C^r, $r \geq 2$, volume-preserving Anosov diffeomorphism. We will assume that $\mathcal{M}$ is endowed with a C^∞ "background" metric which is adapted to the Anosov diffeomorphism. More precisely, there exists a continuous decomposition of the tangent bundle $T\mathcal{M}$ into two invariant subbundles E^u, E^s and numbers $0 < \lambda_s, \lambda_u < 1$ such that for all $n \geq 0$,

$$\begin{aligned} ||df^n(v)|| \leq \lambda_s^n ||v|| &\iff v \in E^s, \\ ||df^{-n}(v)|| \leq \lambda_u^n ||v|| &\iff v \in E^u. \end{aligned} \tag{1}$$

Let $E \subset T\mathcal{M}$ be a subbundle invariant under Df, $\dim E = d \geq 2$. A conformal structure C_x on $E_x \subset T_x\mathcal{M}$ is a class of proportional positive definite quadratic forms on E_x. Using the background metric, we can identify a quadratic form with a symmetric linear operator. Hence, with this identification, a conformal structure is an equivalence class of linear operators. In this class, we can choose a unique representative which has determinant 1 with respect to the background metric. From now on, we understand the conformal structure C_x as this linear operator on E_x.

For each $x \in \mathcal{M}$, we denote the space of conformal structures on E_x by $\mathcal{C}_x = \mathcal{C}_x^E$. Thus we obtain a bundle $\mathcal{C} = \mathcal{C}^E$ over $\mathcal{M}$ whose fiber over x is $\mathcal{C}_x$. A section C of the bundle $\mathcal{C}$ is called a conformal structure on E.

The diffeomorphism f induces a natural pull-back action F on conformal structures as follows. For a conformal structure $C_{fx} \in \mathcal{C}_{fx}$, $F_x(C_{fx}) \in \mathcal{C}_x$ is given by

$$F_x(C_{fx}) = \frac{1}{\det\left((Df_x)^* \circ Df_x\right)} (Df_x)^* \circ C_{fx} \circ Df_x. \tag{2}$$

Here C_{fx} is the linear operator on E_{fx}, and $(Df_x)^* : T_{fx}\mathcal{M} \to T_x\mathcal{M}$ denotes the conjugate operator of Df_x. Clearly, $F_x : \mathcal{C}_{fx} \to \mathcal{C}_x$ is a linear operator.

We say that a conformal structure C is f-invariant if for all $x \in \mathcal{M}$,

$$F_x(C_{fx}) = C_x. \tag{3}$$

Note that the subbundle E can carry an invariant conformal structure only if $E \subset E^s$ or $E \subset E^u$.

We define the norm of a conformal structure C_x as the norm of the quadratic form with respect to the background metric:

$$||C_x|| = \sup_{0 \neq v \in E_x} \frac{< C_x v,\, v >}{||v||^2}. \tag{4}$$

Since the operator C_x is symmetric and positive definite, $||C_x||$ is equal to its largest eigenvalue.

We also define

$$||C_x^{-1}|| = \sup_{0 \neq v \in E_x} \frac{||v||^2}{< C_x v,\, v >}, \tag{5}$$

which is the inverse of the smallest eigenvalue of C_x.

Since the product of all the eigenvalues equals 1, it is easy to see that

$$\begin{aligned} ||C_x^{-1}||^{d-1} \geq ||C_x|| \geq ||C_x^{-1}||^{\frac{1}{d-1}}, \\ ||C_x||^{d-1} \geq ||C_x^{-1}|| \geq ||C_x||^{\frac{1}{d-1}}. \end{aligned} \tag{6}$$

With respect to these norms, we can define L^p spaces of conformal structures. We say that a conformal structure C belongs to L^p if it is measurable and its norm is an L^p function with respect to the invariant volume on $\mathcal{M}$. It is clear that the property that $C \in L^p$ does not depend on the choice of the background metric. The inequalities (6) show that $||C|| \in L^p$ implies $||C^{-1}|| \in L^{p/(d-1)}$ and, similarly, $||C^{-1}|| \in L^p$ implies $||C|| \in L^{p/(d-1)}$.

Conformal structures at two nearby points can be identified as follows. When points x and y are close enough, they can be joined by a unique shortest geodesic. We transport E_x along this geodesic using the Levi-Civita connection, and then project it to E_y. We denote by $S_x^y : \mathcal{C}_x^E \to \mathcal{C}_y^E$ the corresponding identification of conformal structures on E_x and E_y.

If y is in a small neighborhood of x, then the dependence of S_x^y on y is as smooth as the dependence of E_y on the base point. In particular, when E is the stable distribution, S_y^x depends smoothly on y when y moves along the stable manifold.

We now define the $\mathcal{L}_\alpha^{p,(s)}$ spaces for conformal structures. They are a natural extension to manifolds endowed with an Anosov system of similar spaces which are standard in harmonic analysis. We refer to [Ste70] for the results from harmonic analysis that we will need. In this paper we consider dynamical applications of these spaces similar to those in [dlLa].

Definition 2.1. *Let W^s be the stable foliation of f. We say that a homeomorphism $h : \mathcal{M} \to \mathcal{M}$ is adapted to the stable foliation if for any $x \in \mathcal{M}$ we have $h(x) \in W_x^s$.*

Recall that C is a conformal structure on an invariant subbundle E.

Definition 2.2. *We say that $C \in \mathcal{L}_\alpha^{p,(s)}$, $1 \leq p < \infty$, $0 < \alpha \leq 1$, if*

$$\left(\int_{\mathcal{M}} ||C_x - S_{h(x)}^x (C_{h(x)})||^p \right)^{1/p} \leq K \cdot ||h - Id||_{L^\infty}^\alpha \tag{7}$$

for any absolutely continuous homeomorphism h adapted to the stable foliation with $||h - Id||_{L^\infty}$ sufficiently small and $||J_h||_{L^\infty}, ||J_{h^{-1}}||_{L^\infty} < \infty$. Here J_h denotes the Jacobian of the mapping h with respect to the backfround metric.

Analogously, we define the space $\mathcal{L}_\alpha^{p,(u)}$ for the unstable foliation.

We say that $C \in \mathcal{L}_\alpha^p$ if the condition (7) is satisfied for any diffeomorphism h sufficiently close to the identity.

The spaces $\mathcal{L}_\alpha^{p,(s)}$, $\mathcal{L}_\alpha^{p,(u)}$, and $\mathcal{L}_\alpha^p$ are Banach spaces with respect to the norm given by the best possible K in (7).

Our main results are the following two statements.

Theorem 2.1. *Let f be a C^r, $r \geq 2$, volume-preserving Anosov diffeomorphism of a compact manifold $\mathcal{M}$. Let $E \subset T\mathcal{M}$ be an invariant distribution of dimension $d \geq 2$ which is Hölder continuous with exponent $\alpha > 0$.*

Let C be an invariant conformal structure on E such that $||C|| \in L^p$, $p \geq d+1$. Then $C \in \mathcal{L}_\alpha^{p/(d+1)}$.

As a consequence, we obtain the following.

Corollary 2.1. *In addition to the assumptions of Theorem 2.1, suppose that $p > \frac{d \cdot m}{\alpha}$, where $m = dim\, \mathcal{M}$. Then the conformal structure C is continuous.*

For the continuous-time case, we obtain the following analogs of the above statements.

Theorem 2.2. *Let φ^t be a C^r, $r \geq 2$, volume-preserving Anosov flow on a compact manifold $\mathcal{M}$. Let $E \subset T\mathcal{M}$ be an invariant distribution of dimension $d \geq 2$ which is Hölder continuous with exponent $\alpha > 0$.*

Let C be an invariant conformal structure on E such that $||C|| \in L^p$, $p \geq d+1$. Then $C \in \mathcal{L}_\alpha^{p/(d+1)}$.

Corollary 2.2. *In addition to the assumptions of Theorem 2.2, suppose that $p > \frac{d \cdot m}{\alpha}$, where $m = dim\, \mathcal{M}$. Then the conformal structure C is continuous.*

Once it is known that the conformal structure C is continuous, its regularity can be improved as follows. Suppose that the diffeomorphism f (the flow φ^t) is C^∞, and E is tangential to a continuous foliation W with C^∞ leaves, for example, E is the (strong) stable or unstable distribution. Then, the continuity of C implies that C is actually C^∞ along the leaves of the foliation W ([Sad]).

3. Generalizations

The main results can be generalized in some respects.

3.1. We can replace the assumption of f being Anosov by f being partially hyperbolic with uniform accesibility in a measure theoretic sense (see Definition 2.5 of [dlLa]). The only modification needed is using Proposition 2.12 of [dlLa] instead of Proposition 4.1.

3.2. The assumption that f is volume preserving can be weakened to the assumption that $||J_f^\mu - 1||_{L^\infty}$ is sufficiently small, where J_f^μ is the Jacobian of f with respect to a smooth volume μ (see Section 5.3 of [dlLa]).

3.3. In the main statements we assumed that $||C|| \in L^p$, which automatically implies that $||C^{-1}|| \in L^{p/(d-1)}$. If we assume in addition that $||C^{-1}|| \in L^q$ with $q > \frac{p}{d-1}$, the conclusions can be strengthened. For example, if $q = p$, then in in the corollaries we obtain that C is continuous if $p > \frac{2m}{\alpha}$.

4. Some results from Harmonic Analysis

We will use the following results from harmonic analysis, which already were a basic tool in [dlLa].

Proposition 4.1. *For* $1 \leq p \leq \infty$, $0 < \alpha \leq 1$,

$$\mathcal{L}_\alpha^{p,(s)} \cap \mathcal{L}_\alpha^{p,(u)} \subset \mathcal{L}_\alpha^p.$$

We start by observing that any diffeomorphism h close to the identity can be written as $h = h^u \circ h^s$, where h^s, h^u are absolutely continuous homeomorphisms adapted to the stable and unstable foliations respectively. This follows from the absolute continuity of stable and unstable foliations of Anosov diffeomorphisms, and the implicit function theorem (we refer for more details to [dlLa] Proposition 2.6).

Moreover, we have

$$||h^{s,u} - \mathrm{Id}||_{L^\infty} \leq K_1 \cdot ||h - \mathrm{Id}||_{L^\infty},$$

and the Jacobians of h^s and h^u are uniformly bounded.

Then, for any $\Psi \in \mathcal{L}_\alpha^{p,(s)} \cap \mathcal{L}_\alpha^{p,(u)}$, we can estimate

$$\begin{aligned}
||\Psi \circ h - \Psi||_{L^p} &\leq ||\Psi \circ h^u \circ h^s - \Psi \circ h^s||_{L^p} + ||\Psi \circ h^s - \Psi||_{L^p} \\
&\leq K_2 \cdot ||\Psi \circ h^u - \Psi||_{L^p} + ||\Psi \circ h^s - \Psi||_{L^p} \\
&\leq K_2 K_3 \cdot ||h^u - \mathrm{Id}||_{L^\infty}^\alpha + K_3 \cdot ||h^s - \mathrm{Id}||_{L^\infty}^\alpha \\
&\leq K \cdot ||h - \mathrm{Id}||_{L^\infty}.
\end{aligned}$$

□

We denote by W_α^p the potential space $\{f \mid (-\Delta + \mathrm{Id})^{\alpha/2} f \in L^p\}$, i.e. W_α^p is the image of L^p under $(-\Delta + \mathrm{Id})^{-\alpha/2}$. In this paper, we will only consider $1 < p < \infty$. The limiting cases $p = 1, \infty$ are very special.

Proposition 4.2. *Assume that* $0 < \alpha < 1$, $1 < p < \infty$. *Then,*

(a) $\mathcal{L}_\alpha^p \subset W_{\alpha'}^p$ *for any* $\alpha' < \alpha$;

(b) $\mathcal{L}_\alpha^p \subset L^{q-\epsilon}$ *for any* $\epsilon > 0$, *where* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$;

(c) $\mathcal{L}_\alpha^p \subset C^0$ *for* $p > \frac{m}{\alpha}$, *where* $m = \dim \mathcal{M}$.

In all cases, the embeddings are continuous.

Using partitions of unity and coordinate patches, we reduce the proof of Proposition 4.2 to a proof in Euclidean space.

Then, to prove (a), one can use the estimates in [Ste70] §3.3 and in §3.5.2 p. 141. Note that the spaces that we are calling here $\mathcal{L}_\alpha^p$ are called in [Ste70] $\Lambda_\alpha^{p,\infty}$. The potential spaces W_α^p are denoted by another letter in [Ste70].

In some cases, e.g. $p = 2$, the results can be improved slightly, but we will not be concerned with this.

After that, (b) and (c) are consequences of the standard Sobolev embedding theorem. A standard proof for the fractional cases we need is in [Tay97], p. 22 ff.

$\square$

5. Proofs

5.1. Proof of Theorem 2.1. Using invariance of the conformal structure under the powers of F, we rewrite the difference $C_x - S^x_{hx}(C_{hx})$ in a form suitable for making estimates.

Iterating the invariance equation (3) we have

$$C_x = F_x \circ \cdots \circ F_{f^{n-1}x}(C_{f^n x}). \tag{8}$$

Note that, by (2), the mapping

$$F^n_x \equiv F_x \circ \cdots \circ F_{f^{n-1}x} : \; \mathcal{C}_{f^n x} \to \mathcal{C}_x$$

is given by

$$F^n_x(C_{f^n x}) = (A^n_x)^* \circ C_{f^n x} \circ A^n_x, \qquad \text{where} \qquad A^n_x = \frac{1}{\det(Df^n_x)} Df^n_x.$$

When E_x is equipped with the metric given by C_x and $E_{f^n x}$ is equipped with the metric given by $C_{f^n x}$, $A^n_x : \; E_x \to E_{f^n x}$ is an isometry. It follows from (4) and (5) that

$$||(A^n_x)^*|| = ||A^n_x|| \leq \sqrt{||C^{-1}_{f^n x}||} \cdot \sqrt{||C_x||}.$$

and hence

$$||F^n_x|| \leq ||(A^n_x)^*|| \cdot ||A^n_x|| \leq ||C^{-1}_{f^n x}|| \cdot ||C_x||. \tag{9}$$

Considering (8) at the point hx, we have

$$C_{hx} = F_{hx} \circ \cdots \circ F_{f^{n-1}(hx)}(C_{f^n(hx)}).$$

Now we assume that h is adapted to the stable foliation of f so that $f^i(hx)$ is sufficiently close to $f^i(x)$, and thus the space $E_{f^i(hx)}$ can be identified with $E_{f^i x}$. We have

$$S^x_{hx}(C_{hx}) = S^x_{hx} \circ F_{hx} \circ \left(S^{fx}_{f(hx)}\right)^{-1} \circ S^{fx}_{f(hx)} \circ \ldots \tag{10}$$

$$\circ \left(S^{f^{n-1}x}_{f^{n-1}(hx)}\right)^{-1} \circ S^{f^{n-1}x}_{f^{n-1}(hx)} \circ F_{f^{n-1}(hx)} \circ \left(S^{f^n x}_{f^n(hx)}\right)^{-1} \circ S^{f^n x}_{f^n(hx)} \left(C_{f^n(hx)}\right).$$

We denote

$$\tilde{F}_{f^i x} = S^{f^i x}_{f^i(hx)} \circ F_{f^i(hx)} \circ \left(S^{f^{i+1}x}_{f^{i+1}(hx)}\right)^{-1} : \; \mathcal{C}_{f^{i+1}x} \to \mathcal{C}_{f^i x}.$$

and

$$\tilde{F}^i_x = \tilde{F}_x \circ \cdots \circ \tilde{F}_{f^{i-1}x}. \tag{11}$$

Now, we can write (10) as

$$S^x_{hx}(C_{hx}) = \tilde{F}_x \circ \cdots \circ \tilde{F}_{f^{n-1}x} \circ S^{f^n x}_{f^n(hx)}\left(C_{f^n(hx)}\right). \tag{12}$$

Using (8) and (12), we rewrite the difference $C_x - S^x_{hx}(C_{hx})$ by adding and substracting appropriate terms.

$$\begin{aligned}
C_x - S^x_{hx}(C_{hx}) &= F_x \circ \cdots \circ F_{f^{n-1}x}(C_{f^n x}) - \tilde{F}_x \circ \cdots \circ \tilde{F}_{f^{n-1}x} \circ S^{f^n x}_{f^n(hx)}(C_{f^n(hx)}) \\
&= F_x \circ \cdots \circ F_{f^{n-1}x}(C_{f^n x}) - \tilde{F}_x \circ F_{fx} \circ \cdots \circ F_{f^{n-1}x}(C_{f^n x}) \\
&\quad + \tilde{F}_x \circ F_{fx} \circ \cdots \circ F_{f^{n-1}x}(C_{f^n x}) - \tilde{F}_x \circ \cdots \circ \tilde{F}_{f^{n-1}x} \circ S^{f^n x}_{f^n(hx)}(C_{f^n(hx)}) \\
&= (F_x - \tilde{F}_x) \circ F_{fx} \circ \cdots \circ F_{f^{n-1}x}(C_{f^n x}) \\
&\quad + \tilde{F}_x \circ \left(F_{fx} \circ \cdots \circ F_{f^{n-1}x}(C_{f^n x}) - \tilde{F}_{fx} \circ \cdots \circ \tilde{F}_{f^{n-1}x} \circ S^{f^n x}_{f^n(hx)} C_{f^n(hx)})\right) \\
&= \cdots \\
&= \sum_{i=0}^{n-1} \tilde{F}_x \circ \cdots \circ \tilde{F}_{f^{i-1}x} \circ \left(F_{f^i x} - \tilde{F}_{f^i x}\right) \circ F_{f^{i+1}x} \circ \cdots \circ F_{f^{n-1}x}(C_{f^n x}) \\
&\quad + \tilde{F}_x \circ \cdots \circ \tilde{F}_{f^{n-1}x}\left(C_{f^n x} - S^{f^n x}_{f^n(hx)}(C_{f^n(hx)})\right) \\
&= \sum_{i=0}^{n-1} \tilde{F}^i_x \circ \left(F_{f^i x} - \tilde{F}_{f^i x}\right)(C_{f^{i+1}x}) + \tilde{F}^n_x\left(C_{f^n x} - S^{f^n x}_{f^n(hx)}(C_{f^n(hx)})\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
C_x - S^x_{hx}(C_{hx}) = &\sum_{i=0}^{n-1} \tilde{F}^i_x \circ \left(F_{f^i x} - \tilde{F}_{f^i x}\right)(C_{f^{i+1}x}) \\
&+ \tilde{F}^n_x\left(C_{f^n x} - S^{f^n x}_{f^n(hx)}(C_{f^n(hx)})\right).
\end{aligned} \tag{13}$$

Now we proceed as in [dlLa]. First we estimate the general term of the sum in (13). Using (9) we obtain

$$||F^i_{hx}|| \le ||C^{-1}_{f^i(hx)}|| \cdot ||C_{hx}||.$$

This can be viewed as an analog of cancellations in [NT98].

Note that $\tilde{F}^i_x = S^x_{hx} \circ F^i_{hx} \circ \left(S^{f^i x}_{f^i(hx)}\right)^{-1}$. Since h is C^0-close to the identity,

$$||\tilde{F}^i_x|| \le K_1 \cdot ||C^{-1}_{f^i(hx)}|| \cdot ||C_{hx}||. \tag{14}$$

Since the restriction of the derivative of f to E is Hölder continuous, F is also Hölder continuous. Hence,

$$\begin{aligned}
||F_{f^i x} - \tilde{F}_{f^i x}|| &= ||F_{f^i x} - S^{f^i x}_{f^i(hx)} \circ F_{f^i(hx)} \circ \left(S^{f^{i+1}x}_{f^{i+1}(hx)}\right)^{-1}|| \\
&\le K_2 \cdot \mathrm{dist}\left(f^i x, f^i(hx)\right)^\alpha \le K_2 \cdot \left(\lambda^i_s \cdot \mathrm{dist}(x, hx)\right)^\alpha,
\end{aligned}$$

where λ_s is the contraction coefficient in (1) and $\alpha > 0$ is the Hölder exponent.

Therefore, we can estimate the general term in (13) as follows

$$\begin{aligned}&||\tilde{F}_x^i \circ \left(F_{f^i x} - \tilde{F}_{f^i x}\right)(C_{f^{i+1}x})|| \\ &\qquad \le K_3 \cdot ||C^{-1}_{f^i(hx)}|| \cdot ||C_{hx}|| \cdot ||C_{f^{i+1}x}|| \cdot \lambda_s^{i\alpha} \cdot \mathrm{dist}(x, hx)^\alpha.\end{aligned}$$

Now we estimate the $L^{p/(d+1)}$ norm of the general term in (13). Let

$$T_i(x) = \tilde{F}_x^i \circ \left(F_{f^i x} - \tilde{F}_{f^i x}\right)(C_{f^{i+1}x}).$$

Since $||C|| \in L^p$, the equations (6) imply that $||C^{-1}|| \in L^{p/(d-1)}$. Thus, using the Hölder inequality, we obtain

$$\begin{aligned}&||T_i||_{L^{p/(d+1)}} \le K_3 \cdot \lambda_s^{i\alpha} \cdot ||h - \mathrm{Id}||^\alpha_{L^\infty} \\ &\qquad \times \left(\int_{\mathcal{M}} (||C^{-1}_{f^i(hx)}|| \cdot ||C_{hx}|| \cdot ||C_{f^{i+1}x}||)^{p/(d+1)}\right)^{(d+1)/p} \\ &\le K_3 \lambda_s^{i\alpha} \cdot ||h - \mathrm{Id}||^\alpha_{L^\infty} \cdot ||C^{-1}_{f^i(h(\cdot))}||_{L^{p/(d-1)}} \cdot ||C_{h(\cdot)}||_{L^p} \cdot ||C_{f^{i+1}(\cdot)}||_{L^p}\end{aligned} \tag{15}$$

Since f preserves the volume,

$$||C_{f^{i+1}(\cdot)}||_{L^p} = ||C||_{L^p},$$

and since h is a C^1-close to the identity diffeomorphism,

$$||C_{h(\cdot)}||_{L^p} \le K_4 \cdot ||C||_{L^p}, \tag{16}$$

$$||C^{-1}_{f^i(h(\cdot))}||_{L^{p/(d-1)}} = ||C^{-1}_{h(\cdot)}||_{L^{p/(d-1)}} \le K_4 \cdot ||C^{-1}||_{L^{p/(d-1)}}.$$

Hence we obtain that if $C \in L^p$, the general term in (13) is bounded in $L^{p/(d+1)}$ by

$$||T_i||_{L^{p/(d+1)}} \le K_5 \cdot ||h - \mathrm{Id}||^\alpha_{L^\infty} \cdot ||C||^2_{L^p} \cdot ||C^{-1}||_{L^{p/(d-1)}} \cdot \lambda_s^{i\alpha}.$$

Thus,

$$\begin{aligned}&||C_x - S^x_{hx}(C_{hx})||_{L^{p/(d+1)}} \\ &\qquad \le \sum_{i=0}^{n-1} ||T_i||_{L^{p/(d+1)}} + ||\tilde{F}^n_x(C_{f^n x} - S^{f^n x}_{f^n(hx)}(C_{f^n(hx)}))||_{L^{p/(d+1)}} \\ &\qquad \le K \cdot ||h - \mathrm{Id}||^\alpha_{L^\infty} + ||\tilde{F}^n_x(C_{f^n x} - S^{f^n x}_{f^n(hx)}(C_{f^n(hx)}))||_{L^{p/(d+1)}}.\end{aligned}$$

Now we show that the last term in (13) tends to 0 as $n \to \infty$. Let us define linear operators R_n by

$$(R_n(C))_x = \tilde{F}^n_x \left(C_{f^n x} - S^{f^n x}_{f^n(hx)}(C_{f^n(hx)})\right).$$

Then by (14),

$$||(R_n(C))_x|| \le K_1 \cdot ||C^{-1}_{f^n(hx)}|| \cdot ||C_{hx}|| \cdot ||C_{f^n x} - S^{f^n x}_{f^n(hx)}(C_{f^n(hx)})||.$$

Since h is C^1-close to the identity, the norms of the operators $S^{f^n x}_{f^n(hx)}$ are uniformly bounded in x and n. Now, using (16) and the fact that f is volume-preserving, we obtain:

$$\begin{aligned}||C_{f^n x} - S^{f^n x}_{f^n(hx)}(C_{f^n(hx)})||_{L^p} &\le ||C_{f^n x}||_{L^p} + K_6 \cdot ||C_{f^n(hx)}||_{L^p} \\ &\le (1 + K_6 \cdot K_4) \cdot ||C||_{L^p}.\end{aligned}$$

Using Hölder inequality as in (15) it is easy to see that the norms of the linear operators R_n from L^p to $L^{p/(d+1)}$ are bounded uniformly in n. Note that $||(R_n(C))_x||$ tends to 0 uniformly on $\mathcal{M}$ for any continuous structure C. Since the continuous structures are dense in L^p, we conclude that $||(R_n(C))||_{L^{p/(d+1)}} \to 0$ as $n \to \infty$ for any L^p structure C.

Thus we conclude that

$$||C_x - S^x_{hx}(C_{hx})||_{L^{p/(d+1)}} \le K \cdot ||h - \mathrm{Id}||^{\alpha}_{L^\infty},$$

and hence $C \in \mathcal{L}_\alpha^{\frac{p}{d+1},(s)}$.

A similar argument shows that $C \in \mathcal{L}_\alpha^{\frac{p}{d+1},(u)}$. Then Proposition 4.1 implies that $C \in \mathcal{L}_\alpha^{p/(d+1)}$. This completes the proof of Theorem 2.1.

5.2. Proof of Theorem 2.2. The argument here is similar to the proof of Theorem 2.1. We will indicate modifications required for the flow case.

Instead of the stable and unstable foliations we consider the strong stable and strong unstable foliations. We say that a conformal structure C is in $\mathcal{L}_\alpha^{p,(s)}$ (in $\mathcal{L}_\alpha^{p,(u)}$) if the condition (7) is satisfied for any diffeomorphism h adapted to the strong stable (unstable) foliation. The argument in the proof of Theorem 2.1 shows that $C \in \mathcal{L}_\alpha^{\frac{p}{d+1},(s)} \cap \mathcal{L}_\alpha^{\frac{p}{d+1},(u)}$.

We can also define the space $\mathcal{L}_\alpha^{p,(o)}$ by considering the diffeomorphisms adapted to the orbit foliation. For this case, there exists a natural identification of conformal structures on E_x and on $E_{h(x)}$ given by the flow. We use this identification in place of $S^x_{h(x)}$ in the condition (7). With this definition, the difference in (7) is identically 0 for any invariant conformal structure, and the condition is trivially satisfied. Thus,

$$C \in \mathcal{L}_\alpha^{\frac{p}{d+1},(s)} \cap \mathcal{L}_\alpha^{\frac{p}{d+1},(u)} \cap \mathcal{L}_\alpha^{\frac{p}{d+1},(o)}.$$

Now, an analog of Proposition 4.1 shows that $C \in \mathcal{L}_\alpha^{\frac{p}{d+1}}$.

5.3. Proof of Corollaries 2.1 and 2.2. Since the conformal structure C is in L^p, Theorem 2.1 (2.2) together with Proposition 4.2(b) imply that

$$C \in \mathcal{L}_\alpha^{p/(d+1)} \subset L^{q-\epsilon}$$

for any $\epsilon > 0$, where $\frac{1}{q} = \frac{d+1}{p} - \frac{\alpha}{m}$. Calculating q we obtain

$$q = p \cdot \frac{m}{m(d+1) - \alpha p} > p \quad \text{for} \;\; p > \frac{d \cdot m}{\alpha}$$

Since the factor $\frac{m}{m(d+1)-\alpha p}$ increases with p, we can apply the theorem repeatedly until we obtain that $C \in \mathcal{L}^q_\alpha$ with $q \geq \frac{m}{\alpha}$. Then it follows from Proposition 4.2(c) that the conformal structure C is continuous.

□

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References

[dlLa] R. de la Llave. Bootstrap of regularity for integrable solutions of cohomology equations. To appear, Available from `www.ma.utexas.edu` 02-378.

[dlLb] R. de la Llave. Further rigidity properties of conformal Anosov systems. *Erg. Th. Dyn. Syst.* To appear, Available from `www.ma.utexas.edu` 02-418.

[dlL02] R. de la Llave. Rigidity of higher-dimensional conformal Anosov systems. *Ergodic Theory Dynam. Systems*, 22(6):1845–1870, 2002.

[KS03] B. Kalinin and V. Sadovskaya. On local and global rigidity of quasiconformal Anosov diffeomorphisms. *Journal of the Institute of Mathematics of Jussieu*, 2(4):567–582, 2003.

[NP99] M. Nicol and M. Pollicott. Measurable cocycle rigidity for some non-compact groups. *Bull. London Math. Soc.*, 31(5):592–600, 1999.

[NT98] V. Niţică and A. Török. Regularity of the transfer map for cohomologous cocycles. *Ergodic Theory Dynam. Systems*, 18(5):1187–1209, 1998.

[PP97] W. Parry and M. Pollicott. The Livsic cocycle equation for compact Lie group extensions of hyperbolic systems. *J. London Math. Soc. (2)*, 56(2):405–416, 1997.

[Sad] V. Sadovskaya. On uniformly quasi-conformal Anosov systems. *Math. Res. Lett.* To appear, Available from `www.southalabama.edu/mathstat/personal_pages/sadovska/`.

[Ste70] E. Stein. *Singular integrals and differentiability properties of functions.* Princeton University Press, Princeton, N.J., 1970. Princeton Mathematical Series, No. 30.

[Tay97] M. Taylor. *Partial differential equations. III.* Springer-Verlag, New York, 1997. Nonlinear equations, Corrected reprint of the 1996 original.

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