MA 497B Final Exam Review Sheet

Each ticket will contain a theoretical question from the list below and a problem. Questions on these topics may also be asked after you answer the questions on the ticket.

- **Circle rotations**
  - Rational rotations: periods of points
  - Irrational rotations:
    - Minimality: statement and *proof*
    - Equidistribution: statement
    - Distribution of first digits of powers: statement and connection to circle rotations

- **Contractions**
  - Definition
  - Contractions of an interval: statements and *proofs*
  - Contraction Principle: statement and *proof*
  - Contractions in $\mathbb{R}^k$: statement

- **Periodic and fixed points**
  - Definitions;
  - Attracting fixed points: definition,
    a sufficient condition for a fixed point of an interval map: statement and *proof*

- **Non-decreasing continuous interval maps:**
  - Fixed points: statement and *proof*
  - Orbits of points: statement and *proof*

- **Autonomous differential equations,** $\dot{x} = g(x)$
  - Lipschitz functions: definition and examples
  - Behavior of the solutions: statement and *proof*
  - Existence and uniqueness of solutions: statement and *proof*,
    an example of non-uniqueness

- **Linear maps of the plane**
  - "Models"
  - Invariant curves and orbits of points for each "model"
  - Conjugacy of real 2×2 matrices to the models
  - Topological conjugacy of dynamical systems
  - Linear differential equations in the plane and matrix exponential

- **Homeomorphisms**
  - Definition
  - Continuity of the inverse for a continuous bijection: two *proofs*

- **Recurrence**
  - Measurable sets and Lebesgue measure – an overview
  - Measure-preserving map: definition
  - Poincare Recurrence Theorem: statement and *proof*
  - Poincare Recurrence – a topological version: statement and *proof*

- **Topological transitivity, minimality, and topological mixing**
  - Definitions and examples
  - Minimality and closed invariant sets: statement and *proof*
  - Criteria for topological transitivity: four equivalent statements and *proof* of their equivalence
  - Meaning of topological mixing

- **Times-m map of the circle**
  - Fixed and periodic points
  - Writing numbers in base m
  - Constructing a point with dense orbit
  - Topological mixing

- **Sequence spaces Ω_m and Ω_m^+, and the shift map**
  - Definitions
  - Metrics on Ω_m and Ω_m^+, and open balls in these metrics
  - The spaces are compact and do not have isolated points, the shift is continuous
  - Fixed points and periodic points
  - Topological mixing with *proof*
  - Symbolic dynamical system: definition and examples

- **Expanding maps of the circle**
  - Definition of an expanding map
  - Definitions of lift and degree for a circle map
  - For expanding circle maps of degree 2:
    - Fixed points
    - Coding and semiconjugacy with the shift
    - Definition of semiconjugacy for two dynamical systems
    - Topological conjugacy for expanding circle maps of degree 2 (or m) and implications for periodic points, transitivity, etc.

- **Linear maps of the torus $\mathbb{T}^2$**
  - The torus
  - Invertible linear maps (automorphisms) of the torus
  - Hyperbolic toral automorphisms
    - An example, and the action on the fundamental domain (unit square)
    - The eigenvalues are irrational and eigendirections have irrational slopes
    - A point is periodic if and only if its coordinates are rational, with *proof*
    - The number of periodic points of period n: statement and an outline of a proof; a formula for the case if eigenvalues $\lambda > 1$ and $1/\lambda$
    - Topological mixing, with *proof*

- **Topological entropy**
  - Definition and three quantities that can be used in the definition (N, S, D)
  - Topological entropy of isometries and contractions is zero, with *proof*
  - Entropy of E_m is $\log m$, with *proof*
  - If two metrics generate the same topology, the entropy is the same, with *proof*
  - Topological entropy is an invariant of topological conjugacy, with *proof*
  - Properties of topological entropy
  - Topological entropy of a hyperbolic toral automorphism

- **Chaos and Sensitive dependence on the initial conditions**
  - Definitions
  - Examples
  - Chaotic maps exhibit sensitive dependence
  - Topological mixing implies sensitive dependence, with *proof*